Math 539 Homework 9

April 27, 2004, due Thursday May 6

Problem 1: (Calculating \( \pi_n(S^n \vee S^n) \)).
(i) In class we calculated \( \pi_n(S^n \vee S^n), n \geq 2 \), by the following argument: because \( S^n \times S^n = (S^n \vee S^n) \cup e^{2n} \),
\[
\pi_n(S^n \vee S^n) = \pi_n(S^n \times S^n) = \mathbb{Z} \oplus \mathbb{Z}, \quad n \geq 2.
\]
Check all details. You may use the fact that \( \pi_n(S^n) \cong \mathbb{Z} \).
(ii) Here is another argument, that also works for \( n \geq 2 \). Consider the pair \( (X, A) = (S^n \vee S^n, S^n) \) where the sphere \( A \) is one of the obvious factors. Use the Blakers–Massey theorem to show that \( \pi_n(S^n \vee S^n, S^n) \cong \pi_n(S^n) \). Now argue using the exact sequence of the pair \( (S^n \vee S^n, S^n) \). (Remember there are maps from \( S^n \vee S^n \) to both of its factors.)
(iii) Deduce from (i) or (ii) that \( \pi_n(\vee_{j \in A} S^n_j) \cong \oplus_{j \in A} \mathbb{Z} \).

Problem 2: (Symmetric products of wedges of spheres.)
(i) Show that \( SP(X \vee Y) \cong SPX \times SPY \). (Here \( \cong \) means that these spaces are homeomorphic.) You can show this by direct construction.
(ii) For our purposes it is enough to know that the spaces \( SP(X \vee Y) \) and \( SPX \times SPY \) are homotopy equivalent. Prove this by considering the fibration coming from the sequence \( X \rightarrow X \vee Y \rightarrow Y \) etc.
(iii) Deduce from (ii) and Problem 1 that if \( X \) is the wedge product of spheres \( \vee_{j \in A} S^n_j \) then \( \pi_n(X) \rightarrow \pi_n(SP(X)) \) is an \( (n+1) \)-equivalence.

Problem 3: What is \( SP(S^1 \times S^1) \)? Can you work it out using the fact that \( S^1 \times S^1 = (S^1 \vee S^1) \cup e^2 \) and you know \( SP(S^1 \vee S^1) \)?

Problem 4: (Maps to \( K(\pi, n) \)'s) (i) In class we saw that if \( X \) is \( (n-1) \)-connected and
\[
f : \pi_n(X) \rightarrow G = \pi_n(K(G, n))
\]
is any homomorphism then there is a map \( \hat{f} : X \rightarrow K(G, n) \) such that
\[
\hat{f}_* : \pi_n(X) \rightarrow \pi_n(K(G, n))
\]
is \( f \). Show that \( \hat{f} \) is unique up to homotopy.

\textit{Hint:} You have a map \( X \times \{0, 1\} \rightarrow K(G, n) \). Show that it can be extended to \( X \times I \).
(ii) Deduce that if \( X \) is \( (n-1) \)-connected then \( H^n(X; G) \cong \text{Hom}(\pi_n(X), G) \).