Problem 1: (Excision for $\pi_1$) Suppose that $X = A \cup B$ and set $C := A \cap B$. Find the best conditions you can under which the inclusion $\pi_1(A, C) \to \pi_1(X, B)$ is an isomorphism (of pointed sets). Prove your claim. 
*Hint:* Adapt the proof of the Seifert–van Kampen theorem.

Problem 2: (Step 2 in Blakers–Massey theorem.) Let $X$ be a CW complex that is the union of the subcomplexes $A, B$. Set $C := A \cap B$. Suppose that $(A, C)$ is $(m-1)$-connected and that $(B, C)$ is $(n-1)$-connected. The Blakers–Massey thm says that the map $i_* : \pi_q(A, C) \to \pi_q(X, B)$ is an isomorphism for $2 \leq q < m + n - 2$ and an epi for $q = m + n - 2$. We proved this in class when $A = C \cup \text{cells of dim } \leq m$ and $B = C \cup \text{cells of dim } \geq n$. 
*Hint:* You only need to prove this when $A, B$ are obtained by adding a finite number of cells. (Why?) Therefore you can argue by induction on the numbers of added cells. Suppose you obtain $A$ by adding a single cell to $A' \supset C$. Let $X' = A' \cup B$. Then consider the relation of the triads $(X; A, B)$, $(X'; A', B)$ and $(X; A, X')$. The argt is easier when you add cells to $B$.

Problem 3: (Calculating $\pi_n(S^n)$.) There is a homomorphism $\phi : \pi_n(S^n) \to \mathbb{Z}$ given by taking the degree of any smooth map homotopic to $f : S^n \to S^n$. 
(i) Define $\phi$ precisely, show it is well defined. 
(ii) Show that $\phi$ is injective. 
*Hint:* Assume $f$ is smooth, pick a regular value $x$ and then homotop $f$ so that it is “linear” (ie has standard form) in a finite set of disjoint discs centered on the points in $f^{-1}(x)$. Then homotop $f$ so it takes the interiors of these discs onto $S^n \setminus y$ (where $y$ is the antipode of $x$) and takes the rest of $S^n$ to $y$. Then $f$ is a composite $S^n \to S^n \vee \ldots \vee S^n \xrightarrow{g} S^n,$ where the middle space is the one point union of $k$ copies of $S^n$, $k := \#\{f^{-1}(x)\}$. If $f$ has degree 0 then $k = 2\ell$ and you can construct $g$ to be the identity on $\ell$ of the spheres and a reflection on the other $\ell$ spheres. Now show how to homotop such a pair of maps $S^n \vee S^n \to S^n$ to zero. 
Go through the above steps first for $n = 1$ and then for $n = 2$. It would be okay to write out the above proof in the case $n = 2$. 
*Note:* In general I am rather lax in my treatment of base points. But this is permissible. eg if $X$ is simply connected then there is a bijective correspondence between the homotopy classes of based maps $(X, x_0) \to (X, x_0)$ and the homotopy classes of arbitrary (unbased) maps $X \to X$. So when $n > 1$ we need not worry about keeping the base point fixed when calculating $\pi_n(S^n)$. 