Problem 0: I did not state the first problem on last week's homework with enough precision. The correct statement is that for connected manifolds $H^1(M;\mathbb{Z}) \cong [M,S^1]$. If one starts with integral cohomology. The idea: each class in $H^1(M;\mathbb{Z})$ can be represented by a closed 1-form $\alpha$ whose integral round each closed loop in $M$ is an integer. Given such $\alpha$ define $f_\alpha : M \to S^1 = \mathbb{R}/\mathbb{Z}$ that takes the base point $x_0 \in M$ to $0 \in \mathbb{R}/\mathbb{Z}$ by defining $f_\alpha(x) = \int_\gamma \alpha$ where $\gamma$ is any smooth path from $x_0$ to $x$. Now check that $f_\alpha^* (d\theta) = \alpha$. **Hint:** You must calculate the value of $f_\alpha^*(d\theta)$ in the direction $v \in T_x(M)$. Since $f_\alpha(x)$ is independent of the choice of path $\gamma$ you may assume that $\gamma$ is tangent to $v$ at its endpoint $x$.

Problem 1: I promised a homework problem about **basepoints**, good and bad, to help answer the question of when $X$ is homeomorphic to the quotient $X/A$ (for some closed subset $A \subset X$.) There does not seem to be a good general answer to this question. In the cases we are interested in (e.g. $\mathbb{R}^n/\partial \mathbb{R}^n \cong S^n$) the quotient $X/A$ is an $n$-manifold. In this case the base point $x_0$ in $X/A$ has a neighborhood that is homeomorphic to an open $n$-ball and so $X$ is homeomorphic to $X/A$ iff $A$ has a neighborhood $N$ such that $N \setminus A$ is homeomorphic to the annulus $S^{n-1} \times (0,1)$.

There is a notion of a **nondegenerate** base point $x_0$ in a space $X$. Here the condition is that the inclusion $x_0 \hookrightarrow X$ has the HEP. This condition has its uses, but it does not help in the homeomorphism problem. Here are some questions.

(i) Suppose that $A$ is a closed subset of $X$ with the HEP. Show that the base point $x_0 \in X/A$ also has the HEP.

(ii) Find an example of a closed contractible subset $A \subset X$ such that the base point in $X/A$ is nondegenerate but $X$ is not homeomorphic to $X/A$.

**NOTE:** the next defn is slightly changed from what I said in lecture: it is probably better to stick to the language in the text book.

A **map** $j : A \to X$ is called a **cofibration** if given any homotopy $F : A \times I \to Z$ and any map $f : X \to Z$ that extends $F(\cdot,0) : A \to Z$ in the sense that $f \circ j = F(\cdot,0)$, the map $f$ is the time $0$ map of a homotopy $\tilde{F} : X \times I \to Z$ such that $\tilde{F}(j(a),t) = F(a,t)$ for all $a \in A, t \in [0,1]$. (If we are in the based category all maps are assumed to preserve the base point.)

A **pair** $(X,A)$ is said to have the HEP (**Homotopy Extension Property**) if the inclusion $j : A \to X$ is a cofibration. It follows from Ex 3 below that a map $j : A \to X$ is a cofibration iff $j$ is a homeomorphism onto its image and the pair $(X,j(A))$ has the HEP.

**Problem 2** (The Hopf map) Think of $S^3$ as the unit sphere in $\mathbb{C}^2$:

$$S^3 = \{ (z_1,z_2) : |z_1|^2 + |z_2|^2 = 1 \}.$$

Define $f : S^3 \to S^2 = \mathbb{C} \cup \{ \infty \}$ by $f(z_1,z_2) = z_1/z_2$.

(i) Show that the inverse image of each point in $S^2$ is a circle in $S^3$. 

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(ii) Show that the complex projective plane may be decomposed as $S^2 \cup_f B^4$ where $f: \partial B^4 = S^3 \to S^2$ is the Hopf map.

**Hint:** Think of $\mathbb{C}P^2$ as the space of all complex lines through 0 in $\mathbb{C}^3$. The subset of lines that intersect the plane $z_3 = 1$ forms an open subset $U$ of $\mathbb{C}P^2$ whose complement can be identified with $S^2 := \mathbb{C}P^1$, “the line at infinity”. Identify $U$ with the interior of the (real) 4-ball $B^4 \subset \mathbb{R}^4$ in such a way that you see the attaching map is $f$. As a warmup, it is probably a good idea to do the real projective plane $\mathbb{R}P^2$ and the complex projective line $S^2 = \mathbb{C}P^1 = \text{pt} \cup B^2$.

**Problem 3** (More on the HEP and cofibrations) (i) Show that if $j: A \to X$ is a cofibration then $j: A \to j(A)$ is a homeomorphism (where $j(A)$ is given the subspace topology). i.e. we can think of $j$ as the inclusion of a subset $A$ of $X$ into $X$. (Hint: take $Z$ to be the mapping cylinder or the mapping cone of $j$.)

(ii) Let $A \subset X$ be closed. Show that $(X, A)$ has the HEP iff $W := X \times \{0\} \cup A \times I$ is a retract of $X \times I$, i.e. there is a map $r : X \times I \to W$ that is the identity on $W \subset X \times I$.

(iii) Show that if $X$ is normal and $A$ is closed, the pair $(X, A)$ has the HEP iff there is a neighborhood $V$ of $A$ in $X$ such that $(V, A)$ has the HEP. (i.e in this case having the HEP is a local property for $A$, depending only on a neighborhood of $A$ in $X$.) *Recall: $X$ is normal* if any two closed sets can be separated by disjoint open sets. The relevant property is given by Urysohn’s lemma.

**NOTE** There is an interesting class of metric spaces called ANRs (ANR= Absolute Neighborhood Retract) with the property that if $X$ and $A$ are ANRs such that $A$ is a closed subset of $X$ then $(X, A)$ has the HEP. Every finite dimensional manifold and every paracompact manifold modelled on a Banach space is an ANR. I won’t have time to go into this, but this is often a useful technical condition.

**Problem 4** (More on mapping cones) The first part spells out what it means for the mapping cone to be a “natural” construction; the second part shows that its homotopy type only depends on the “homotopy class” of $f: X \to Y$.

Let $\mathcal{C}$ be the category whose objects are morphisms $f: X \to Y$ in the category $\mathcal{T}_*$ of based top spaces, and whose morphisms are commutative diagrams

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow g_X & & \downarrow g_Y \\
X' & \xrightarrow{f'} & Y'.
\end{array}
$$

More formally, $\text{Mor}_\mathcal{C}((X, Y, f), (X', Y', f'))$ is the set of pairs $(g_X, g_Y)$ that make the diagram commute, where $g_X \in \text{Mor}_{\mathcal{T}_*}(X, X')$ and $g_Y \in \text{Mor}_{\mathcal{T}_*}(Y, Y')$.

(i) Show that the mapping cone $f \sim C_f$ is a functor from $\mathcal{C}$ to $\mathcal{T}_*$.

(ii) Show that if the morphisms $g_X, g_Y$ are homotopy equivalences then $C_f$ is homotopy equivalent to $C_{f'}$. 
