Math 539 Homework 1
January 30, 2004

I recorrected Q 2, which can be proved in its original form if you use (i).

**Problem 1.** (i) Show that if $X$ is a Hausdorff topological space then every compact subset of $X$ is closed.

(ii) Show that if $X$ is locally compact and Hausdorff (i.e. $X$ is the union of open subsets with compact closures) then for every open $U$ and point $x \in U$ there is an open set $V$ with compact closure such that $x \in V \subset \overline{V} \subset U$.

**Problem 2.** (i) Suppose that $X$ is Hausdorff, let $K \subset X$ be compact and suppose that $K \subset U_1 \cup \cdots \cup U_n$ where the $U_i$ are open. Show that $K$ is the union of a finite number of compact sets $K_j$, $j = 1, \ldots, M$, such that each $K_j$ is contained in some $U_i$.

(ii) Consider the iterated mapping space $M(X, M(Y, Z))$ where $X$ is Hausdorff and $Y$ is locally compact and Hausdorff. Let $K \subset X$ be compact, $L_1, L_2 \subset Y$ be compact and $U_1, U_2 \subset Z$ be open. Denote by $U^L$ the set of maps $f : Y \to Z$ such that $f(L) \subset U$. Show that 
\[
\left( U_1^{L_1} \cup U_2^{L_2} \right)^K
\]
is a finite intersection of sets of the form $(U^L)^K$. Hence deduce that the sets $(U^L)^K$ (with $U$ open and $K, L$ compact) form a subbasis for the compact–open topology in $M(X, M(Y, Z))$.
(We assume that $M(Y, Z)$ is also given the compact-open topology.)

(iii) Show that the sets $U^{K \times L}$ form a subbasis for the topology on $M(X \times Y, Z)$ where $U$ (resp. $K, L$) ranges over all open (resp. compact) subsets of $Z$ (resp. $X, Y$).

(iv) Deduce that the map $\phi : M(X \times Y, Z) \to M(X, M(Y, Z))$ is a homeomorphism, where for each $g \in M(X \times Y, Z)$
\[
\phi(g)(x) : y \mapsto g(x, y).
\]

**Problem 3** (i) If $A$ is a subspace of $X$ and $B$ is a subspace of $Y$ we denote by $M(X, A; Y, B)$ the subset of $M(X, Y)$ consisting of maps $f : X \to Y$ such that $f(A) \subset B$. We give it the subspace topology. Suppose that $B$ is a single point $\ast$ in $Y$. Then there is an obvious bijection
\[
\phi : M(X, A; Y, \ast) \to M(X/A, \ast; Y, \ast)
\]
where $X/A$ denotes the quotient space with base point $\ast$ equal to the image of $A$ in $X/A$. Show that if $A$ is compact this is a homeomorphism.

(ii) Let $X$ be the unit ball $B$ in $\mathbb{R}^n$ and $A$ be its boundary $\partial B = S^{n-1}$. Show that the quotient space $X/A$ is homeomorphic to the sphere $S^n$ (where you define $S^n$ as the unit sphere in $\mathbb{R}^{n+1}$).