This problem set concerns the following concepts.

Let $A$ be a subset of $\mathbb{R}$. A point $c \in \mathbb{R}$ is called a boundary point of $A$ if every $\epsilon$-neighborhood of $c$ contains a point of $A$ and a point of its complement $\mathbb{R} \setminus A$. (Note that $c$ does NOT have to be in $A$.)

A point $c \in A$ is called an interior point of $A$ if there is $\epsilon > 0$ such that the $\epsilon$-neighborhood of $x$ is entirely contained in $A$.

The subset $A$ of $\mathbb{R}$ is said to be open if for every $x \in A$ there is $\epsilon > 0$ such that the $\epsilon$-neighborhood of $x$ is entirely contained in $A$.

The subset $B$ of $\mathbb{R}$ is said to be closed iff its complement $\mathbb{R} \setminus B$ is open.

**Problem 1.** Let $A = [1, 3)$ (a half open interval).

(i) Show that the points 1 and 3 are boundary points of $A$.

(ii) Show that every point $x \in (1, 3)$ is an interior point of $A$.

(iii) Which of the following sets are closed?

$A = [2, 3)$, $B = \{0, 1/n : n \geq 1\}$, $C = (2, 4)$.

**Problem 2.** Let $B$ be any subset of $\mathbb{R}$. Show that each point $x \in B$ is either a boundary point of $B$ or an interior point of $B$, but cannot be both.

**Problem 3.** Let $B$ be any subset of $\mathbb{R}$.

(i) Show that if $c \in B$ is a boundary point of $B$ then it is a cluster point for the complement $\mathbb{R} \setminus B$ of $B$.

(ii) Prove that if $c$ is a cluster point both for $B$ and for its complement $\mathbb{R} \setminus B$ then $c$ is a boundary point of $B$.

(iii) Take $B = \{1, 2\}$, a set containing just 2 points. What are its boundary points? What are its cluster points?

**NOTE:** The first version of Question 3(i) was wrong: it is not true that a boundary point $c$ of $B$ has to be a cluster point of $B$. The trouble is that $c$ might be an isolated point of $B$, ie. there might be $\epsilon > 0$ such that $(c - \epsilon, c + \epsilon) \cap B = \{c\}$. Then $c$ would be a boundary point but not a cluster point.

**Problem 4.** Let $A \subset \mathbb{R}$ be open.

(i) Show that every point of $A$ is an interior point of $A$.

(ii) Use the result of Problem 2 to show that $A$ contains NONE of its boundary points.

The next two parts are **bonus problems.**

(iii) Problem 4(ii) implies that all the boundary points of $A$ lie in its complement $B := \mathbb{R} \setminus A$. Use this together with the definition of a closed set to show that a set is closed iff it contains all its boundary points.

(iv) Finally combine what you have just proved (Problem 4(iii)) with Problem 3(ii) to deduce that a set is closed iff it contains all its cluster points.