Math 319/320 Worksheet 3

Problem 1. (i) *Construct a function $f : \mathbb{N} \to \mathbb{N}$ that is injective but not surjective.*

Define $f(k) = 2k$.

(ii) *Show that if S is an infinite set then there is a function $f : S \to S$ that is injective but not surjective. You may use the fact that S is infinite if and only if there is an injection $\mathbb{N} \to S$.*

Since S is infinite there is an injection $g : \mathbb{N} \to S$. Let $A = g(\mathbb{N})$ be its image. Then $S = A \cup (S \setminus A)$. Then $g : \mathbb{N} \to A$ is bijective and so has an inverse $g^{-1} : A \to \mathbb{N}$. Let $f : \mathbb{N} \to \mathbb{N}$ be as in (i). Then the composite $g \circ f \circ g^{-1} : A \to A$ is well defined, and injective (since f is) but not surjective.

Now define $h : S \to S$ as follows.

If $x \notin A$ define $h(x) = x$.

If $x \in A = f(\mathbb{N})$ define $h(x) = g \circ f \circ g^{-1}(x)$.

Problem 2. (i) *Consider the set $\mathbb{N} \times \mathbb{N}$ of all ordered pairs (p, q) of integers. Show that it is denumerable, i.e. show how to construct a bijection $\mathbb{N} \to \mathbb{N} \times \mathbb{N}$.*

We shall obtain a bijection $\mathbb{N} \to \mathbb{N} \times \mathbb{N}$ by constructing an enumeration of the ordered pairs (p, q). To do this, use the diagonal process; ie think of the ordered pairs (p, q) as points in the plane, and then count along the successive diagonals $p + q = k$ for $k = 2, 3, 4, \ldots$. If we count from left to right along these diagonals then the enumeration starts as

$$(1, 1), (1, 2), (2, 1), (1, 3), (2, 2), (3, 1), (1, 4), \ldots.$$

(ii) *Show that if A and B are denumerable, disjoint sets, then $A \cup B$ is denumerable.*

By assumption there are bijections $f : \mathbb{N} \to A, g : \mathbb{N} \to B$. Define $h : \mathbb{N} \to A \cup B$ by

$$h(2k - 1) = f(k), k \geq 1, \quad h(2k) = g(k), k \geq 1.$$

Then h is obviously surjective. It is injective because A and B are disjoint. More formally, suppose that $h(x) = h(y)$ but $x \neq y$. Then x, y cannot both be odd since f is injective and they cannot both be even since g is injective. But if one (say x) is odd and the other is even, then $h(x) \in A$ equals $h(y) \in B$. Hence $A \cap B$ is nonempty, contrary to hypothesis.

Problem 3. *Let a, b, c, d be real numbers which satisfy $0 < a < b < c < d$.*

a) Is it true that $bc < ad$? If it is true prove the inequality. If it is not true give an example of four real numbers which violate the inequality.

FALSE: Take $b = 1, c = 2, d = 3$ and $a = 1/4$.

b) *Is it true that $ca < bd$? If it is true prove the inequality. If it is not true give an example of four real numbers which violate the inequality.*

FALSE: take $a = 1, b = 2, c = 3$ and $d = 100$.

c) *Assume that $0 < c^2 < c$ for some real number c. Show that $0 < c < 1$.*

If $c = 1$ then $c = c^2$. So the inequality $c^2 < c$ is not satisfied. If $c > 1$ then $c = 1+a$ where $a = c-1 > 0$. Hence $c^2 = (1+a)^2 = 1+2a+a^2 > 1+a = c$. So the inequality $c^2 < c$ is also not satisfied. Hence, by the trichotomy rule, the only possibility left is that $c < 1$. Since $c > 0$ by assumption and the transitivity rule ($c > c^2$ and $c^2 > 0$ implies $c > 0$), we find $0 < c < 1$.

Problem 4. a) *Show that if x and y are rational numbers then $x + y$ and xy are rational numbers.*

To say x, y are rational means that there are integers p, q, r, s (where $q \neq 0, s \neq 0$) such that $x = p/q, y = r/s$. Then $x + y = (ps + qr)/qs$ and $xy = pr/qs$ and retional.

b) *If $x \neq 0$ is rational and y irrational, show that xy is irrational.*

Argue by contradiction. By assumption $x = p/q$ where $p \neq 0$. Therefore $x^{-1} = 1/x = q/p$ is also a rational number. If xy were rational then $xy(x^{-1}) = y$ would also be rational by (a). But y is irrational by assumption. Therefore xy is irrational.

c) *If x and y are irrational, is it always true that $x + y$ is irrational? Explain.*

NO: Suppose that x is irrational and let $y = 1 - x$. Then y is not rational, since if it were $-y$ would be rational and hence $-y + 1 = x$ would be rational. On the other hand, $x + y = 1$ is rational. (ie the irrationalities of x, y can cancel.)