Math 319/320 Worksheet 3

Problem 1. (i) Construct a function \( f : \mathbb{N} \to \mathbb{N} \) that is injective but not surjective.

Define \( f(k) = 2k \).

(ii) Show that if \( S \) is an infinite set then there is a function \( f : S \to S \) that is injective
but not surjective. You may use the fact that \( S \) is infinite if and only if there is an
injection \( \mathbb{N} \to S \).

Since \( S \) is infinite there is an injection \( g : \mathbb{N} \to S \). Let \( A = g(\mathbb{N}) \) be its image. Then
\( S = A \cup (S \setminus A) \). Then \( g : \mathbb{N} \to A \) is bijective and so has an inverse \( g^{-1} : A \to \mathbb{N} \). Let
\( f : \mathbb{N} \to \mathbb{N} \) be as in (i). Then the composite \( g \circ f \circ g^{-1} : A \to A \) is well defined, and
injective (since \( f \) is) but not surjective.

Now define \( h : S \to S \) as follows.
If \( x \notin A \) define \( h(x) = x \).
If \( x \in A = f(\mathbb{N}) \) define \( h(x) = g \circ f \circ g^{-1}(x) \).

Problem 2. (i) Consider the set \( \mathbb{N} \times \mathbb{N} \) of all ordered pairs \((p, q)\) of integers. Show
that it is denumerable, i.e. show how to construct a bijection \( \mathbb{N} \to \mathbb{N} \times \mathbb{N} \).

We shall obtain a bijection \( \mathbb{N} \to \mathbb{N} \times \mathbb{N} \) by constructing an enumeration of the
ordered pairs \((p, q)\). To do this, use the diagonal process; i.e. think of the ordered pairs
\((p, q)\) as points in the plane, and then count along the successive diagonals \( p + q = k \)
for \( k = 2, 3, 4, \ldots \). If we count from left to right along these diagonals then the
enumeration starts as

\[
(1, 1), (1, 2), (2, 1), (1, 3), (2, 2), (3, 1), (1, 4), \ldots.
\]

(ii) Show that if \( A \) and \( B \) are denumerable, disjoint sets, then \( A \cup B \) is denumerable.

By assumption there are bijections \( f : \mathbb{N} \to A, g : \mathbb{N} \to B \). Define \( h : \mathbb{N} \to A \cup B \)
by

\[
h(2k - 1) = f(k), k \geq 1, \quad h(2k) = g(k), k \geq 1.
\]

Then \( h \) is obviously surjective. It is injective because \( A \) and \( B \) are disjoint. More
formally, suppose that \( h(x) = h(y) \) but \( x \neq y \). Then \( x, y \) cannot both be odd since
\( f \) is injective and they cannot both be even since \( g \) is injective. But if one (say \( x \)) is
odd and the other is even, then \( h(x) \in A \) equals \( h(y) \in B \). Hence \( A \cap B \) is nonempty,
contrary to hypothesis.

Problem 3. Let \( a, b, c, d \) be real numbers which satisfy \( 0 < a < b < c < d \).

a) Is it true that \( bc < ad \)? If it is true prove the inequality. If it is not true give an
example of four real numbers which violate the inequality.

False: Take \( b = 1, c = 2, d = 3 \) and \( a = 1/4 \).

b) Is it true that \( ca < bd \)? If it is true prove the inequality. If it is not true give an
example of four real numbers which violate the inequality.

False: Take \( a = 1, b = 2, c = 3 \) and \( d = 100 \).

c) Assume that \( 0 < c^2 < c \) for some real number \( c \). Show that \( 0 < c < 1 \).
If \( c = 1 \) then \( c = c^2 \). So the inequality \( c^2 < c \) is not satisfied. If \( c > 1 \) then \( c = 1 + a \) where \( a = c - 1 > 0 \). Hence \( c^2 = (1 + a)^2 = 1 + 2a + a^2 > 1 + a = c \). So the inequality \( c^2 < c \) is also not satisfied. Hence, by the trichotomy rule, the only possibility left is that \( c < 1 \). Since \( c > 0 \) by assumption and the transitivity rule (\( c > c^2 \) and \( c^2 > 0 \) implies \( c > 0 \)), we find \( 0 < c < 1 \).

**Problem 4.** a) Show that if \( x \) and \( y \) are rational numbers then \( x + y \) and \( xy \) are rational numbers.

To say \( x, y \) are rational means that there are integers \( p, q, r, s \) (where \( q \neq 0, s \neq 0 \)) such that \( x = p/q, y = r/s \). Then \( x + y = (ps + qr)/qs \) and \( xy = pr/qs \) and rational.

b) If \( x \neq 0 \) is rational and \( y \) irrational, show that \( xy \) is irrational.

Argue by contradiction. By assumption \( x = p/q \) where \( p \neq 0 \). Therefore \( x^{-1} = 1/x = q/p \) is also a rational number. If \( xy \) were rational then \( xy(x^{-1}) = y \) would also be rational by (a). But \( y \) is irrational by assumption. Therefore \( xy \) is irrational.

c) If \( x \) and \( y \) are irrational, is it always true that \( x + y \) is irrational? Explain.

NO: Suppose that \( x \) is irrational and let \( y = 1 - x \). Then \( y \) is not rational, since if it were \(-y\) would be rational and hence \(-y + 1 = x\) would be rational. On the other hand, \( x + y = 1 \) is rational. (ie the irrationalities of \( x, y \) can cancel.)