Problem 1. Fill in the blanks in the following proof that 
\[ A \cup (B \cap C) = (A \cup B) \cap (A \cup C). \]

If \( x \in A \cup (B \cap C) \) then either \( x \in A \) or \( x \in B \cap C \). If \( x \in A \) then \( x \in A \cup B \) and \( x \in A \cup C \) and so \( x \in (A \cup B) \cap (A \cup C) \). On the other hand, if \( x \in B \cap C \) then \( x \in B \) and \( x \in C \). Hence \( A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C) \).

Now suppose that \( x \in (A \cup B) \cap (A \cup C) \). Then \( x \in A \cup B \) and \( x \in A \cup C \).

If \( x \in A \) then: \( x \in A \cup (B \cap C) \).

On the other hand if \( x \notin A \) then \( x \in B \) (because \( x \in A \cup B \)) and \( x \in C \) (because \( x \in A \cup C \)).

Therefore \( x \) is either in \( A \) or in \( B \cap C \), i.e. \( x \in A \cup (B \cap C) \).

Problem 2. It is possible to take intersections and unions of many sets \( A_i, i \in I \), not just two. We define

\[ \bigcup_{i \in I} A_i := \{ x : \exists i \in I \text{ such that } x \in A_i \}, \quad \bigcap_{i \in I} A_i := \{ x : x \in A_i \forall i \in I \}. \]

The set \( I \) is called the indexing set. Often it is the set of the first \( n \) integers \( \{1, \ldots, n\} \), but sometimes it is the infinite set \( \mathbb{N} \) of all positive integers.

(i) Find three subsets \( A_1, A_2, A_3 \) of the plane \( \mathbb{R}^2 \) such that each double intersection \( A_i \cap A_j \) is nonempty but the triple intersection \( A_1 \cap A_2 \cap A_3 \) is empty.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Here I wrote \( A_{12} \) to mean \( A_1 \cap A_2 \), etc.}
\end{figure}

(ii) Find open intervals \( A_i = (a_i, b_i) \subset \mathbb{R} \) such that each finite intersection \( \cap_{1 \leq i \leq n} A_i \) is nonempty but the infinite intersection \( \cap_{i \in \mathbb{N}} A_i \) is empty.

Take for example, \( A_i = (0, 1/i) \).
Problem 3. Let $f : A \rightarrow B$ be a function and $C \subset A, D \subset B$. Show that $C \subseteq f^{-1}(f(C))$ and $f(f^{-1}D) \subseteq D$.

$f^{-1}(f(C)) = \{a \in A : f(a) \in f(C)\}$. If $x \in C$ then $f(x) \in f(C)$ by defn of $f(C)$, hence $x$ satisfies the condition to be in $f^{-1}(f(C))$.

(In words: $f^{-1}(f(C))$ is the set of all points whose image is contained in the image $f(C)$ of $C$. But obviously the points in $C$ have image in $f(V)$.)

If $x \in f^{-1}(D)$ then $f(x) \in D$ by defn of the inverse image. Hence $f(f^{-1}D) \subseteq D$.

If $f$ is injective, do either of these inclusions become equalities?

$f$ is injective iff $f(x) = f(y)$ implies $x = y$. To say $f(x) \in f(C)$, means that there is $c \in C$ such that $f(x) = f(c)$ (by defn of the set $f(C)$.) Hence if $f$ is injective $x$ must equal $c$. Since this holds for all $x$ such that $f(x) \in f(C)$, $f^{-1}(f(C)) = C$.

But the second statement about $D$ won’t hold unless the image of $f$ contains $D$, and you can only be sure of this when $f$ is surjective.

(eg take $f : [0, \infty) \rightarrow \mathbb{R}, x \mapsto x$ and $D = (-2, -1)$.)

What if $f$ is surjective? Now the first statement need not hold, but the second will. You should find examples here on your own.

Problem 4. Let $A, B$ be subsets of a universal set $U$. Simplify the following expressions. You can draw Venn diagrams to help you. (i) $(A \cap B) \cup (U \setminus A)$ and (ii) $A \cup [B \cap (U \setminus A)]$.

(i) $(A \cap B) \cup (U \setminus A) = B \cup (U \setminus A)$.

Proof: Since $A \cap B \subset B$, $(A \cap B) \cup (U \setminus A) = B \cup (U \setminus A)$.

Now suppose $x \in B \cup (U \setminus A)$. If $x \in U \setminus A$ then $x \in (A \cap B) \cup (U \setminus A)$, as required. So we need to consider the case when $x \notin U \setminus A$. This means that $x \in A$. Since $x \in B \cup (U \setminus A)$, in this case $x$ must be in $B$. Hence $x \in A \cap B$. So $x$ does lie in $(A \cap B) \cup (U \setminus A)$.

(ii) $A \cup [B \cap (U \setminus A)] = A \cup B$. Here again it is obvious that LHS $\subseteq$ RHS (where LHS = left hand side means the set $A \cup [B \cap (U \setminus A)]$ and RHS = right hand side means $A \cup B$. To show RHS $\subseteq$ LHS we only need to consider the case $x \notin A$ (since if $x \in A$ it is obvious.) But if $x \notin A$ and $x \in B$ then $x \in U \setminus A$ and $x \in B$, i.e. $x \in B \cap (U \setminus A) \subseteq$ LHS.