Math 319/320 Worksheet 2

Problem 1. Fill in the blanks in the following proof that
\[ A \cup (B \cap C) = (A \cup B) \cap (A \cup C). \]

If \( x \in A \cup (B \cap C) \) then either \( x \in A \) or \( x \in B \cap C \). If \( x \in A \) then \( x \in A \cup B \) and \( x \in \) \hspace{1cm} and so \( x \in \) \hspace{1cm}. On the other hand, if \( x \in B \cap C \) then \( x \in \) \hspace{1cm} and \( x \in \) \hspace{1cm}. Hence \( A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C) \).

Now suppose that \( x \in \) \hspace{1cm}. Then \( x \in \) \hspace{1cm} and \( x \in \) \hspace{1cm}. On the other hand if \( x \notin A \) then

Therefore

Problem 2. It is possible to take intersections and unions of many sets \( A_i, i \in I \), not just two. We define
\[ \cup_{i \in I} A_i := \{ x : \exists i \in I \text{ such that } x \in A_i \}, \quad \cap_{i \in I} A_i := \{ x : x \in A_i \forall i \in I \}. \]
The set \( I \) is called the indexing set. Often it is the set of the first \( n \) integers \( \{1, \ldots, n\} \), but sometimes it is the infinite set \( \mathbb{N} \) of all positive integers.

(i) Find three subsets \( A_1, A_2, A_3 \) of the plane \( \mathbb{R}^2 \) such that each double intersection \( A_i \cap A_j \) is nonempty but the triple intersection \( A_1 \cap A_2 \cap A_3 \) is empty.

(ii) Find open intervals \( A_i = (a_i, b_i) \subset \mathbb{R} \) such that each finite intersection \( \cap_{1 \leq i \leq n} A_i \) is nonempty but the infinite intersection \( \cap_{i \in \mathbb{N}} A_i \) is empty.
Problem 3. Let $f : A \to B$ be a function and $C \subset A, D \subset B$. Show that $C \subset f^{-1}(f(C))$ and $f(f^{-1}D) \subset D$.

If $f$ is injective, do either of these inclusions become equalities?

What if $f$ is surjective?

Problem 4. Let $A, B$ be subsets of a universal set $U$. Simplify the following expressions. You can draw Venn diagrams to help you.

(i) $(A \cap B) \cup (U \setminus A)$

(ii) $A \cup [B \cap (U \setminus A)]$