Math 319: Suggested projects.

November 15 2005

Here are some possible topics for your project. I would like you to work in pairs (or possibly in threes) on this project. Each group will make a short oral presentation during the last two weeks of the semester highlighting one or two of the main arguments. (Each group should divide the topic into separate parts, so that each member of the group contributes their own bit of the presentation: there should be 5 minutes each.) Each person will also individually write a 6-8 page paper on the topic chosen. (This can be handwritten (especially the formulas etc) but must be legible.) I am most interested in your constructing valid arguments and explaining the mathematics clearly, though you can give a small amount of background. You should outline the whole project but concentrate your calculations on your special part. The essays should describe specific examples, not just be general and vague.

Different groups should work on different projects. So as soon as your group have decided what you want to do, please tell me. Email me at *dusa@math.sunysb.edu.* Give alternatives in case another group has already taken your favorite topic. If you cannot find a suitable group to work with, email me anyway with details of what you would like to do and I will try to put you in touch with people to work with.

Timetable for the rest of the semester. This is quite tight; it will be hard to fit everything in, partly because I am out of town Dec 3-11.

1. I will spend time on Nov 15 and 17 describing the projects, but will also start lecturing on continuous functions Ch 5.1 and Ch 5.3. I will continue lecturing on this subject on Nov 22, Nov 29 and Dec 1. There will be one homework, due Dec 8.
2. The last 20 minutes of the Nov 22 class will be a retest of proof-writing. See the class web page for further details.
3. You must tell me your project title by Nov 22 at the very latest. If you want to choose a different topic from those outlined below, please give me a detailed outline by Nov 22 (earlier if possible.)
4. The first draft of the paper is due **Friday Dec 2.** The final draft is due **Tuesday Dec 13.**
5. I will have extended office hours during the week Nov 30–Dec 2 to discuss the projects with you. I expect you all will have questions: all the problems below require thought.
6. I will be out of town Dec. 3-11, but Tanvir will be available to help you. There will be no class on Tuesday Dec 6; Tanvir will have read the first drafts of the projects and will have comments by then. (If you can get your first draft to me by Thursday Dec 1, I will try to give you feedback before I go away.)
7. Your presentations will be given during class times on **Thursday Dec 8, Monday Dec 12** (at 2:20) and **Tuesday Dec 13.** Attendance will be taken and will form a small part of the grade for the project. Prof Phillips will be in charge of the class on Dec 8.

Projects on Sequences

Project 1: Square roots. Example 3.3.5 gives a sequence that calculates $\sqrt{a}$. This sequence is derived from Newton's method of finding roots of an equation. (You can find this in any first year Calculus text.) Explain this. Use the algorithm to calculate $\sqrt{5}$ correct to 3 decimal places. How many terms do you need? Explain the error estimate. There is another way finding square roots by a long division process, a method that "every school boy used to learn" — perhaps 100 years ago. Find out about this and explain it. Is it based on the same or a different algorithm?

Project 2: The Fibonacci sequence. This is the well known sequence $1, 1, 2, 3, 5, 8, \ldots$ defined inductively by the equation $s_{n+1} = s_n + s_{n-1}$. It turns out that the ratios of successive terms

$r_n = \dfrac{s_{n+1}}{s_n}$ form a sequence that is eventually monotonic and converges to the number known as the Golden Ratio (the positive root of the equation $x^2 = x + 1$.) Use the method of Ex 3.3.5 to explain this. Explore the effect of starting with pairs of different numbers, eg with $1, 5$ or with $-2, 1$. Do these sequences also converge? If so, what are their limits?

Project 3: Contractive sequences Instead of the rule $s_{n+1} = s_n + s_{n-1}$, consider the sequence defined by $s_{n+1} = as_n + (1-a)s_{n-1}$, where $0 < a < 1$. Thus s_{n+1} is a weighted average of s_n and s_{n-1}. This sequence is contractive and converges. The case $a = 1/2$ is explained on p 82/3 in Ex 3.5.6(a). You could discuss a different case (say $a = 1/4$ or $a = 1/3$.) Contractive sequence are also used in Ex 3.5.10 to find roots of certain polynomials. Use this method to find a root of the equation $x^4 + 2x - 2 = 0$ lying between 0 and 1 correct to 5 decimal places. (This is a good project for three people; one could explain contractive sequences and the other two could do the different examples.

Project 4: A divergent sequence. Investigate the sequence (x_n) where $x_n = \sin n$. Show how to construct a subsequence of (x_n) that converges to $+1$ and another that converges to -1. What properties of the sine function and the number π do you use in your argument? Can you show that this sequence has a subsequence that sconverges to any given number between -1 and 1? It may be helpful to look at Example 3.4.6(c) where it is shown that this sequence is divergent.

Projects on series

Project 5: Binary and ternary Decimals. Find the binary and ternary representations of the numbers $3/8, 1/3, 2/7$, and 32. Explain (using the Completeness Axiom) why any infinite sequence $S = .10011100111\dots$ of zeros and ones represents a *unique* real number x_S. Show how to get a geometric series from the ternary representation of 3/8 and sum it to 3/8. Is it true that S is eventually periodic iff x_S is rational? Explain. This project is based on the end of Ch 2.5. See also Ex. 3.7.2 (a) (geometric series). It is somewhat related to Project 8.

Project 6: Variations on the harmonic series. If $\{m_1, m_2, \dots\}$ is the collection of natural numbers that end in 6 then $\sum_{k\geq 1} \frac{1}{m_k}$ diverges. (*Hint:* adapt the proof that the harmonic series $\sum_{n\geq 1} \frac{1}{n}$ is divergent.) The next result is more unexpected. Let $\{n_1, n_2, \dots\}$ be the collection of natural numbers that do NOT use the digit 6 in their decimal expansion. (ie 345145 is in this collection while 3456247 is not.) Show that $\sum_{k\geq 1} \frac{1}{n_k}$ converges to a number less than 80. What can you say about $\sum_{k\geq 1} \frac{1}{p_k}$ if $\{p_1, p_2, \dots\}$ be the collection of natural numbers that do not involve 4 in their decimal expansion. *Note:* This is an adaptation of ex 16 on p 263. You will have to read about the convergence of sequences of positive numbers from Section 3.7 and Ch 9, specially Ex. 3.7.2 (a) (geometric series), 3.7.6(b) (harmonic series) and the Comparison test 3.7.7

Project 7: The harmonic series and rearrangements. Show how to rearrange the terms of the alternating harmonic series $\sum_{n\geq 1} (-1)^{n+1}\frac{1}{n} = 1 - 1/2 + 1/3 - 1/4 + 1/5 - ..$ so that the series (a) converges to 2, and (b) diverges. (In a rearrangement you permute the order of the terms but not their signs, eg you might consider $1 + 1/3 - 1/2 + 1/5 + 1/7 - 1/4 \dots$) Also discuss other patterns of signs in the harmonic series. eg can you decide if $1 - 1/2 - 1/3 + 1/4 - 1/5 - 1/6 + 1/7 - \dots$ converges? (Here signs are $+, -, -, +, -, -, + \dots$.) What about $1 - 1/2 - 1/3 + 1/4 + 1/5 + 1/6 - 1/7 - \dots$? (Here there is one $+$, then two $-$, then three $+$ then four $-$ and so on.) What about one $+$, then two $-$, then four $+$ then eight $-$ and so on, ie. where you use powers of 2? *Note:* This is an example of a conditionally convergent sequence, ie. the corresponding sequence of positive terms does not converge. You will find relevant definitions on p 89, and p 255. cf also Ex 3.3.3(b), Ex 3.7.6(b).

Projects on Sets

Project 8: The Cantor set $\mathbb{F}$. This is a paradoxical subset of $[0, 1]$ discovered by Cantor. It is obtained from the unit interval by first removing the open interval $I_1 = (1/3, 2/3)$ (which leaves two intervals each of length $1/3$), then removing the middle thirds of these two intervals (leaving four intervals each of length $1/9$), then removing the middle thirds of these four intervals which gives you 8 intervals of length $1/27$ and so on.... See p 316–318. Give a detailed description of this set $\mathbb{F}$ as an infinite intersection. Show it is closed (cf Def 11.1.2) and that all its points are cluster points. Explain its relation to ternary decimals, ie decimals to base 3. Show that its complement in $[0, 1]$ is the union of infinitely many disjoint open intervals of total length 1. So you might think you'd taken away all the points in $[0, 1]$. But in fact $\mathbb{F}$ has uncountably many points. Explain clearly and in detail why this is so.

Project 9: Countable and Uncountable sets. Give a variety of examples of Countable and Uncountable sets. For example, show that the set of sequences (x_n) where $x_n = 0, 1,$ or 2 is uncountably infinite. Show that its subset consisting of sequences with only a finite number of nonzero entries is countably infinite. What about the set of sequences that are eventually periodic i.e. they are periodic if you ignore a finite set of terms at the beginning?

More theoretical projects

Uniform continuity and convergence are the most important concepts covered in MAT 320 and not in MAT 319. Projects 11 and 12 use the Weierstrass M-test (on p 268), which involves uniform convergence.

Project 10: Uniform continuity: Give examples of continuous functions that are not uniformly continuous. Explain why any continuous function defined on the interval $[a, b]$ is uniformly continuous. This concept is important when defining the Riemann integral: explain.

Project 11: Uniform convergence: Explain why the uniform limit of continuous functions is continuous (p 235). Give an example of a sequence of continuous functions that converges pointwise to a discontinous limit. Explain the Weierstrass M-test.

The next two projects concern examples whose discovery astounded the mathematics community of the day. There are many possible references for these examples, and you can use them if you prefer. The main thing is to describe the examples carefully and then explain why they have the properties claimed.

Project 12: A continuous nowhere differentiable function. Give a careful explanation of the example on p 354. Why is your function not differentiable at the point $1/\pi$? (You will need to understand the definition of the derivative from 6.1.1.)

Project 13: A space filling curve. Explain the example on p 355 — or you might be able to find another one online. Explain why your curve goes through an arbitrary point in the square, say $(1/\sqrt{2}, 1/sqrt3)$. Draw some pictures, using a computer graphics system if possible. This is related to fractal curves, snowflakes etc.

Various other projects

Project 14: The fundamental theorem of algebra. This says that any polynomial with complex coefficients has a complex root. Find a proof and explain it. How does your proof apply in the case of the polynomial $x^4 + 4x + 4$? (Why do you know this polynomial has no real roots?)

Project 15: Card Tricks. Martin Gardner has a lovely article in the *College Math Journal* (2000), 173–177, about card tricks. Pick one of the tricks and explain it fully. eg the "nonmessing up theorem" on p 173 or the curiosity on the bottom of p 175. Or you could do one of the challenge problems at the end.