Math 319 Solutions to Homework 8

Problem 1. (a) *Give an example of a sequence that is bounded above but not bounded below and that has a convergent subsequence.*

There are many examples. eg $x_n = 1$ if n is even and $x_n = -n$ if n is odd.

(b) *Explain how to construct a monotonic increasing sequence of rational numbers that converges to $\sqrt{3}$.*

Let $x_1 = 1$. We will construct an increasing sequence of rational numbers x_n so that $\sqrt{3} - 1/n < x_n < \sqrt{3}$ for all n. Then $\lim x_k = \sqrt{3}$. Since

$$\sqrt{3} - 1 < 2 - 1 = 1 = x_1,$$

x_1 satisfies the requirements. Suppose that $x_k \in \mathbb{Q}$ has been chosen so that $x_{k-1} \le x_k < \sqrt{3}$ and $\sqrt{3} - 1/k < x_k$. Then choose x_{k+1} to be rational number in the interval

$$\big(\max(x_k, \sqrt{3} - 1/(k+1)), \sqrt{3}\big).$$

Problem 2. *Consider the sequences $x_n = \sin\left(\frac{n\pi}{4}\right)$, $y_n = \frac{1}{\sqrt{n}}$, and $z_n = x_n y_n$.*

(a) *Write down the first 10 terms of (x_n). Find two monotonic subsequences of (x_n) with different limits.*

$$x_1 = \frac{1}{\sqrt{2}} = x_9,\ x_2 = 1 = x_{10},\ x_3 = \frac{1}{\sqrt{2}},\ x_4 = 0,\ x_5 = -\frac{1}{\sqrt{2}},\ x_6 = -1, x_7 = -\frac{1}{\sqrt{2}},\ x_8 = 0.$$

This sequence is periodic with period 8 (i.e. $x_{n+8} = x_n$ for all n.) The only monotonic subsequences are eventually constant. eg $x_1, x_4, x_8, x_{3k}, k \ge 3$ is monotonic decreasing with limit 0. $y_k := x_{6+8k}$ is monotonic with limit -1.

(b) *Which terms in the product sequence (z_n) are peaks?*

Note that (y_n) is strictly decreasing, converging to 0. Therefore the terms z_n with $x_n = 1$ are peaks. These are the terms with $n = 2 + 8k$, for $k \ge 0$. Are there any other peaks? The only other possible candidates have $x_n = \frac{1}{\sqrt{2}}$. i.e. $n = 1 + 8k$ or $3 + 8k$ for some $k \ge 0$.

If $n = 1 + 8k$ then $z_n = \frac{1}{\sqrt{2(1+8k)}}$ and $z_{n+1} = \frac{1}{\sqrt{2+8k}}$. Since $8k + 2 < 2(8k + 1)$ for all $k > 0$, we have $z_n < z_{n+1}$ for these k. Hence z_n is not a peak if $k > 0$. But $z_1 = z_2$ is a peak.

If $n = 3 + 8k$ then $z_n = \frac{1}{(\sqrt{2(3+8k)}}$ and the next positive terms are $z_{n+6} = \frac{1}{(\sqrt{2(9+8k)}} < z_n$ and $z_{n+7} = \frac{1}{(\sqrt{(10+8k)}}$. Since $z_{n+8} < z_n$, z_n is a peak iff it is $\ge z_{n+7}$. This happens iff $10 + 8k \ge 2(3 + 8k)$, that is iff $k = 0$. Hence z_3 is the only peak of this form.

Thus, the peaks are z_1, z_2, z_3 and the terms $z_{2+8k}, k > 0$.

Find a subsequence of (z_n) that is strictly decreasing (i.e. $z_{n_{k+1}} < z_{n_k}$ for all k), and another that is strictly increasing.

A strictly decreasing sequence would be a product $x_n y_n$ with x_n constant and positive. eg $z_{2+8k}, k \ge 0$.

A strictly increasing sequence would be a product $x_n y_n$ with x_n constant and negative. eg $z_{6+8k}, k \ge 0$.

(c) *Does* (z_n) *converge?*

Yes it converges to 0 since $|x_n| \leq 1$ for all n. Hence $|z_n| \leq |y_n| \to 0$. (This uses Thm 3.1.10.)

Problem 3. (a) *Suppose that* $x_n \geq 0$ *for all* n *and that* $\lim x_n = 2$. *Find a subsequence of* $((-1)^n x_n)$ *that converges to* 2 *and another that converges to* -2.

The subsequence $x_{2k}, k \geq 1$, converges to 2 and $x_{2k+1}, k \geq 0$ converges to -2.

Does $((-1)^n x_n)$ *converge?* No, because it has two subsequences with different limits. (cf. 3.4.5)

(b) *Suppose that* $x_n \geq 0$ *for all* n *and you are told that* $((-1)^n x_n)$ *converges. Show that* (x_n) *converges. What is its limit?*

Let $(y_n) := ((-1)^n x_n)$ have limit y. Then its subsequence (y_{2k}) also has limit y (by 3.4.2). Since $y_{2k} = x_k \geq 0$ for all k, we have $y \geq 0$ by 3.2.4. Similarly y_{2k+1} converges to y, and because its terms are all ≤ 0 we find that $y \leq 0$. Hence $y = 0$. Then $|x_n| \leq |y_n| \to 0$ also, by Thm 3.1.10.

Problem 4. *Describe all cluster points of* A *where*

a) $A = \mathbb{Z} \cup (0, 1)$,

The cluster points are $[0, 1]$.

Proof: if $c \in [0, 1]$ then $(c - \delta, c + \delta)$ obviously intersects $[0, 1] \setminus \{c\}$ for all $\delta > 0$. Now suppose that $c \in \mathbb{Z}, c \neq 0, 1$. Then $(c - 1/2, c + 1/2)$ meets A only in the point c. Hence c is not a cluster point. The last case is when $c \notin A, c \neq 0, 1$. Then $\delta := \min\{|c - n| : n \in \mathbb{Z}\}$ is a positive number (it is the minimum distance of c from an integer). And $(c - \delta, c + \delta) \cap A = \emptyset$.

b) $A = \{1/n : n \in \mathbb{N}\}$.

In this case the only cluster point is 0. I won't say more now, since it is like (i) and we discussed it in class today.

Problem 5. (a) *Consider the intervals* $I_k = (1 + \frac{1}{3k+1}, 1 + \frac{1}{3k})$ *for* $k = 1, 2, 3, 4, \ldots$. *Write down* I_1, I_2 *and* I_3 *explicitly.*

$I_1 = (5/4, 4/3)$, $I_2 = (8/7, 7/6)$, $I_3 = (11/10, 10/9)$

(b) *Make an accurate sketch of the set* $A = \cup_{k \geq 1} I_k$.

left to you – these intervals are all disjoint and get closer to 1.

(c) *Describe all cluster points of* A.

The cluster points are 1 together with the union of the closed intervals $[1 + \frac{1}{3k+1}, 1 + \frac{1}{3k}]$.