Math 319 Homework 7
Due Thursday, October 27, 2005

Problem 1. Suppose that \( (x_n) \) is a sequence with limit 3. Prove carefully from the definition that
\[ \lim_{n \to \infty} \frac{1}{x_n} = \frac{1}{3}. \]

Note: I do NOT want you to quote Theorem 3.2.3(b). Instead, adapt the proof to this case.

Problem 2. Suppose that the sequences \((x_n)\) and \((z_n)\) both converge to \(w\) and that \(x_n \leq y_n \leq z_n\) for all \(n\).

(i) Use Theorem 2.1.10 to show that the sequence \((y_n - x_n)\) converges to 0.

(ii) Use (i) and the sum theorem (3.2.3(a)) to conclude that \((y_n)\) converges to \(w\).

Note This is known as the Squeeze Theorem. The book gives a different (more direct) proof in 3.2.7.

Problem 3. Find the limits, justifying your answer carefully. (Use any theorems you like, but say what you are using.)

(i) \[ x_n = \frac{4n^2 - n + 1}{n^2 - 3n}. \]

(ii) \[ (-1)^n \frac{n + 1}{n^2 + 2}. \]

Problem 4. (i) Show that the sequence \(x_n = \frac{(n^2 - n)}{(n + 1)}\) is monotonic.

(ii) Define \(x_n\) inductively by the relation \(x_{n+1} = \frac{1}{2}(x_n + 5/x_n)\). Assume that \(x_n \geq x_{n+1} > 0\) for all \(n\). Then \((x_n)\) converges by Theorem 3.3.2. What is its limit?

Problem 5. Consider the sequence defined inductively by: \(x_{n+1} = 4 - 3/x_n\).

(i) Show that if \(1 < x_1 < 3\) then this is monotonic increasing. What is the limit?

(ii) Show that if \(x_1 > 3\) then this is monotonic decreasing. What is the limit?

(iii) Is there a value of \(x_1\) that gives a sequence with limit 1?

Note The points 1 and 3 are special because they are the roots of the quadratic equation \(x^2 - 4x + 3 = 0\).