Math 319/320 Solutions to Homework 4

Problem 1. *Let* $I_n := [0, \frac{1}{n}]$ *and* $J_n := (0, \frac{1}{n}]$, *for* $n \in \mathbb{N}$. *Show that* $\cap_{n=1}^{\infty} I_n = \{0\}$.

Since $0 \in I_n$ for all n, $0 \in \cap_{n=1}^{\infty} I_n$. If $x \in I_n$ for all n then $0 \leq x \leq 1/n$ for all n. But if $x > 0$ then $1/x > 0$ and there is by Archimedes' principle an integer k such that $k > 1/x$. This means $x > 1/k$. Hence $x \notin I_k$. Hence $x \notin \cap_{n=1}^{\infty} I_n$. Thus the only point in this intersection is 0.

Show $\cap_{n=1}^{\infty} J_n = \emptyset$. Since $J_n \subset I_n$ for all n, $\cap_{n=1}^{\infty} J_n \subset \cap_{n=1}^{\infty} I_n = \{0\}$. But $0 \notin J_1$. Hence $0 \notin \cap_{n=1}^{\infty} J_n$. Thus this intersection is empty.

Problem 2. *Let* A *and* B *be bounded nonempty subsets of* $\mathbb{R}$. *Define* $A + B := \{a + b : \ a \in A, b \in B\}$. *Show that* $\sup(A + B) = \sup A + \sup B$.

Since A, B are bounded and nonempty these sets have suprema. Let $s = \sup A$ and $t = \sup B$. Then $s \geq a, \forall a \in A$ and $t \geq b, \forall b \in B$. Hence $s + t \geq a + b, \forall a \in A, b \in B$. Hence $s + t$ is an upper bound for $A + B$.

To see it is the least upper bound, it suffices to show that for all $\epsilon > 0$ there is $c \in A + B$ such that $c > s + t - \epsilon$. But because $s = \sup A$ there is $a \in A$ such that $a > s - \epsilon/2$. Similarly, there is $b \in B$ such that $b > t - \epsilon/2$. Then $c = a + b \in A + B$ and $c > s + t - \epsilon$.