Math 319/320 Homework 3
Due Thursday, September 22, 2005
revised version

Problem 1. Show that
\[ \left( \frac{1}{2}(a+b) \right)^2 \leq \frac{1}{2}(a^2 + b^2) \]
for all \( a, b \in \mathbb{R} \). Show that equality holds if and only if \( a = b \).

We must show that \( \frac{1}{4}(a^2 + 2ab + b^2) \leq \frac{1}{2}(a^2 + b^2) \). Multiplying both sides by 4, we see this is equivalent to \( a^2 + 2ab + b^2 \leq 2a^2 + 2b^2 \) and hence to \( 0 \leq a^2 + b^2 - 2ab \). But this last inequality holds since \( a^2 + b^2 - 2ab = (a-b)^2 \) and \( x^2 \geq 0 \) for all \( x \). (We proved last on the last HW.) This shows that the inequality holds. Also it shows that we have equality if and only if \( 0 = a^2 + b^2 - 2ab \), i.e. \( (a-b)^2 = 0 \). But this holds iff \( a = b \). (Again this was proved in last HW.)

Problem 2. Assume that \( a < x < b \) and \( a < y < b \). Show that \( |x-y| \leq b-a \). Find a geometric explanation for the obtained inequality.

First proof If \( x = y \) the inequality is obvious since \( b-a > 0 \) by hypothesis. Now assume that \( x < y \). We know that \( x < b \). Also \( a < y \) implies \(-y < -a\). Adding we get
\[ |x-y| = x - y = x + (-y) < b + (-a) = b-a. \]
Similarly, if \( y < x \) we may reverse the roles of \( x \) and \( y \) to find: \( |x-y| = y-x < b-a \). Hence in all cases \( |x-y| \leq b-a \). (in fact we have < here.)

Second proof (which I learnt as I was correcting your HW; it’s essentially the same but slicker.)
By hypothesis \( a < x < b \) and \( a < y < b \). Multiply the second inequality by \(-1\) to get \(-b < -y < -a\). Then add this to the first inequality to get \( a-b < x-y < b-a \). This has the form \(-C < Z < C\) where \( Z = x-y \) and \( C = b-a > 0 \). Hence it is equivalent to \( |Z| < C\), i.e. \( |x-y| < b-a \).

\(|x-y|\) is the distance between \( x \) and \( y \). So geometrically we are saying that the distance between any two points in the interval \((a,b)\) is at most \( b-a \).

Problem 3. Let \( a, b \in \mathbb{R} \) and \( a \neq b \). Show that there exist \( \epsilon \)-neighborhood \( U_{\epsilon}(a) \) of \( a \) and \( \epsilon \)-neighborhood \( V_{\epsilon}(b) \) of \( b \) such that \( U_{\epsilon}(a) \cap V_{\epsilon}(b) \neq \emptyset \).

The problem here is to show you can choose \( \epsilon \) large enough that these sets do intersect. So you must specify \( \epsilon \).

Proof 1 By renaming \( a, b \) we may suppose that \( a < b \). Choose \( \epsilon = 2(b-a) \), twice the distance between \( a \) and \( b \). Then \( b \in U_{\epsilon}(a) = (a-\epsilon, a+\epsilon) \). This is geometrically
obvious, since \( U_\epsilon(a) \) contains all points whose distance from \( a \) is \(< \epsilon \) and \( b \) has distance \( b - a < 2(b - a) = \epsilon \) from \( a \). Since \( b \in U_\epsilon(b) \) for any \( \epsilon \), \( b \) is in the intersection.

But to show it in formulas, note that

\[
U_\epsilon(a) = (a - 2b + 2a, a + 2b - 2a) = (3a - 2b, 2b - a).
\]

We need to see that \( 3a - 2b < b < 2b - a \). But \( 3a < 3b \) implies \( 3a - 2b < b \); while \( b < 2b - a = b + (b - a) \) since \( b - a > 0 \).

As some of you noticed, any \( \epsilon > |b - a|/2 \) will do, since then the average \( (a + b)/2 \) will lie in the intersection. Here is a nice argument to show this:

**Proof 2:** As above, we may suppose that \( a < b \). Note that \( V_\epsilon(a) = (a - \epsilon, a + \epsilon) \) and \( V_\epsilon(b) = (b - \epsilon, b + \epsilon) \). Since \( a < b, a - \epsilon < b \), and the only way to have an overlap of these intervals is for \( a + \epsilon > b - \epsilon \). (You draw this.) i.e. we need \( 2\epsilon > b - a \) or \( \epsilon > (b - a)/2 \). If \( \epsilon \) satisfies this inequality then the average \( (a + b)/2 \) is in both intervals.

**Problem 4.** Let \( S := \{x \in \mathbb{R} : x \geq 0\} \). Show that \( S \) has lower bounds, but no upper bounds. Show that \( \inf S = 0 \).

Clearly 0 is a lower bound for \( S \). Moreover if \( y > 0 \) then \( y \) is not a lower bound for \( S \) since \( y \) is not \( \geq \) the element \( 0 \in S \). Hence every lower bound for \( S \) is \( \leq 0 \). Hence 0 is the greatest lower bound, i.e. \( 0 = \inf S \).

**To show \( S \) has no upper bounds:** **Proof 1**

Since every \( n \in \mathbb{N} \) is positive and so \( > 0 \), then \( \mathbb{N} \subset S \). If \( y \) were an upper bound for \( S \), we would have \( y \geq n \) for all \( n \in \mathbb{N} \), in contradiction to Archimedes' Principle. Hence \( S \) has no upper bounds.

**Proof 2:** Suppose that \( u \) is an upper bound for \( S \). Then \( u \in \mathbb{R} \) and \( u \geq 0 \), since \( 0 \in S \). Also \( u + 1 > u \) (this is true for all real numbers.) Hence \( u + 1 \in \mathbb{R} \) and \( u + 1 > u \geq 0 \). Hence \( u + 1 \in S \). Hence \( u \geq u + 1 \). But this is impossible, by the trichotomy rule. (we cannot have both \( u + 1 > u \) and \( u \geq u + 1 \).) Hence there is no upper bound.

Some of you combined the two arguments above, but it is simpler (and hence better) to use one OR the other.

**Problem 5.** If \( S \subset \mathbb{R} \) contains one of its upper bounds, then this upper bound is the supremum of \( S \).

Let \( y \) be an upper bound for \( S \) and suppose that \( y \in S \). We must show that no \( z < y \) is an upper bound for \( S \). But if \( z \) is an upper bound for \( S \), then \( z \geq y \) since \( y \in S \) and \( z \) is \( \geq \) every element in \( S \). Therefore \( z \) cannot also be \( < y \). Hence \( y \) is the least upper bound for \( S \), i.e. it is the supremum of \( S \).

Note: this argument is almost the same as the proof in ex. 4 that \( 0 = \inf S \).