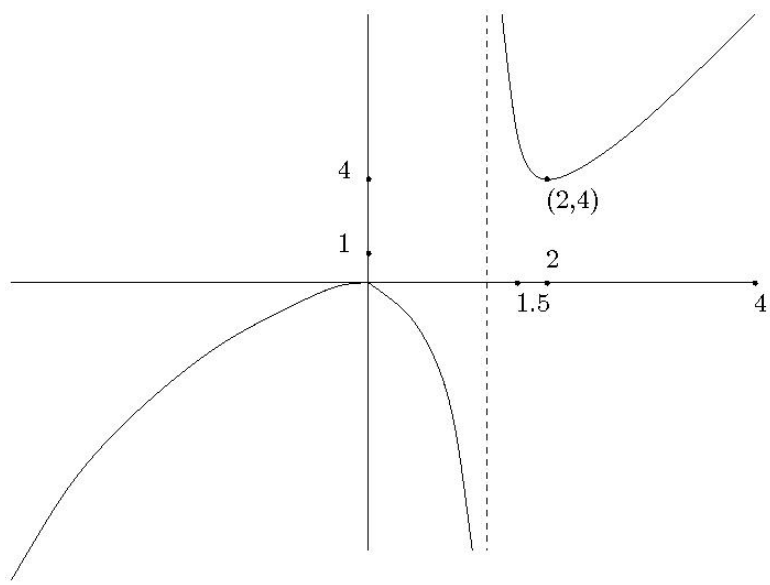
Math 319/320 Solutions to Homework 1

Problem 1. *Prove the identity:* $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

1. To show $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$:
let $x \in A \cup (B \cap C)$. Then $x \in A$ or $x \in B \cap C$. If $x \in A$ then $x \in A \cup B$ and $A \cup C$, Hence $x \in (A \cup B) \cap (A \cup C)$. On the other hand, if $x \in B \cap C$ then x is in both B and C and hence in both $(A \cup B)$ and $(A \cup C)$. Therefore, $x \in (A \cup B) \cap (A \cup C)$.

2. To show $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$:
let $x \in (A \cup B) \cap (A \cup C)$. Then x is in $A \cup B$ and in $A \cup C$. In particular, x is in either A or B. If $x \in A$ then $x \in A \cup (B \cap C)$, as required. If $x \notin A$ then $x \in B$. Since $x \in A \cup C$ we must also have $x \in C$. Hence $x \in B \cap C$. Hence again $x \in A \cup (B \cap C)$.

Problem 2. *Consider the function* $f : \mathbb{R} \setminus \{1\} \to \mathbb{R}$ *given by* $f(x) = x^2/(x-1)$.
(i) Graph it.



(ii) *Find* $f(A)$ *where* A *is the interval* $(1.5, 4)$.
f has a local minimum at 2, and $f(2) = 4$. So answer is $[4, 16/3)$.
(iii) *Find* $f^{-1}(B)$ *where* $B = [1, 4]$. This is the single point 2.
(iv) *Find two subsets* C, D *of* $\mathbb{R} \setminus \{1\}$ *such that* $f(C) \cap f(D) \neq f(C \cap D)$. You could take C, D to be any two different points with the same image. For ex., $f(3) = 9/2$. There is a point $a \in (1, 2)$ that also satisfies the equation $a^2/(a-1) = 9/2$. So take $C = \{a\}, D = \{3\}$.

Problem 3. *Let $f : A \to B$ be a function and suppose that $C \subseteq A$ and $D \subseteq B$. Are the following statements true or false (for every choice of f, C, D)? Justify your answers by a brief proof or a counterexample.*

General comments: As we saw in class the inverse image seems to behave better than the direct image in matters of this sort. So you should be suspicious of (i) – it is more likely that (ii) holds.

(i) $f(A \setminus C) \subseteq f(A) \setminus f(C)$.

This is FALSE (since f need not be injective). eg take $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$. Let $A = \{-2\}, C = \{2\}$. Then $A \setminus C = \{-2\}$ and so $f(A \setminus C) = \{4\}$. But $f(A) = f(C)$ so $f(A) \setminus f(C) = \emptyset$.

(ii) $f^{-1}(B \setminus D) = f^{-1}(B) \setminus f^{-1}(D)$.

This is TRUE. $x \in f^{-1}(B \setminus D)$ iff $f(x) \in B \setminus D$ iff $f(x) \in B$ and $f(x) \notin D$ iff $x \in f^{-1}(B)$ and $x \notin f^{-1}(D)$, that is $x \in f^{-1}(B) \setminus f^{-1}(D)$.

NOTE: here iff = if and only if (a useful shorthand)

Problem 4. *Suppose that $f : A \to B$ and $g : B \to C$ are functions such that the composite $g \circ f$ is surjective. Is g necessarily surjective? What about f? Give brief proofs or counterexamples.*

g must be surjective. Proof: since $g \circ f$ is surjective, for every $c \in C$ there is $a \in A$ such that $g \circ f(a) = c$. But $g \circ f(a) = g(f(a))$. Hence there is an element $b \in B$ such that $g(b) = c$, namely $b = f(a)$.

But f need not be surjective because g need not be injective. eg if $f : [0, \infty) \to \mathbb{R}$ is $f(x) = x$, f is not surjective. Define $g : \mathbb{R} \to [0, \infty)$ by $g(y) = y^2$. Then $g \circ f : [0, \infty) \to [0, \infty)$ is surjective.

Problem 5. Prove by mathematical induction: $3^{2n} - 1$ is divisible by 8 for all $n \geq 1$.

The statement $P(n)$ is: 8 divides the integer $3^{2n} - 1$.

Base case: if $n = 1$ then $P(1)$ says that 8 divides $3^2 - 1 = 8$, which is true.

Inductive step: suppose that 8 divides $3^{2k} - 1$. We must show that 8 divides $3^{2k+2} - 1$. But

$$3^{2k+2} - 1 = 3^2 \times 3^{2k} - 1 = 9(3^{2k} - 1) + 9 - 1 = 9(3^{2k} - 1) + 8.$$

Since 8 divides $3^{2k} - 1$ by the inductive hypothesis, it divides $9(3^{2k} - 1)$. It also divides 8. Hence it divides $9(3^{2k} - 1) + 8 = 3^{2k+2} - 1$, as required.