Problem 1. Prove the identity: \( A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \).

1. To show \( A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C) \):
   - Let \( x \in A \cup (B \cap C) \). Then \( x \in A \) or \( x \in B \cap C \). If \( x \in A \) then \( x \in A \cup B \) and \( A \cup C \), hence \( x \in (A \cup B) \cap (A \cup C) \). On the other hand, if \( x \in B \cap C \) then \( x \) is in both \( B \) and \( C \) and hence in both \( (A \cup B) \) and \( (A \cup C) \). Therefore, \( x \in (A \cup B) \cap (A \cup C) \).

2. To show \( (A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C) \):
   - Let \( x \in (A \cup B) \cap (A \cup C) \). Then \( x \) is in \( A \cup B \) and in \( A \cup C \). In particular, \( x \) is in either \( A \) or \( B \). If \( x \in A \) then \( x \in A \cup (B \cap C) \), as required. If \( x \notin A \) then \( x \in B \). Since \( x \in A \cup C \) we must also have \( x \in C \). Hence \( x \in B \cap C \). Hence again \( x \in A \cup (B \cap C) \).

Problem 2. Consider the function \( f : \mathbb{R} \setminus \{1\} \to \mathbb{R} \) given by \( f(x) = \frac{x^2}{x - 1} \).

(i) Graph it.

(ii) Find \( f(A) \) where \( A \) is the interval \((1.5, 4)\).

\( f \) has a local minimum at 2, and \( f(2) = 4 \). So answer is \([4, 16/3]\).

(iii) Find \( f^{-1}(B) \) where \( B = [1, 4] \). This is the single point 2.

(iv) Find two subsets \( C, D \) of \( \mathbb{R} \setminus \{1\} \) such that \( f(C) \cap f(D) \neq f(C \cap D) \). You could take \( C, D \) to be any two different points with the same image. For ex., \( f(3) = 9/2 \). There is a point \( a \in (1, 2) \) that also satisfies the equation \( a^2/(a - 1) = 9/2 \). So take \( C = \{a\}, D = \{3\} \).
Problem 3. Let \( f : A \to B \) be a function and suppose that \( C \subseteq A \) and \( D \subseteq B \). Are the following statements true or false (for every choice of \( f, C, D \))? Justify your answers by a brief proof or a counterexample.

General comments: As we saw in class the inverse image seems to behave better in matters of this sort. So you should be suspicious of (i) – it is more likely that (ii) holds.

(i) \( f(A \setminus C) \subseteq f(A) \setminus f(C) \).
This is FALSE (since \( f \) need not be injective). eg take \( f : \mathbb{R} \to \mathbb{R} \) given by \( f(x) = x^2 \). Let \( A = \{-2\}, C = \{2\} \). Then \( A \setminus C = \{-2\} \) and so \( f(A \setminus C) = \{4\} \). But \( f(A) = f(C) \) so \( f(A) \setminus f(C) = \emptyset \).

(ii) \( f^{-1}(B \setminus D) = f^{-1}(B) \setminus f^{-1}(D) \).
This is TRUE. \( x \in f^{-1}(B \setminus D) \) iff \( f(x) \in B \setminus D \) iff \( f(x) \in B \) and \( f(x) \notin D \) iff \( x \in f^{-1}(B) \) and \( x \notin f^{-1}(D) \), that is \( x \in f^{-1}(B) \setminus f^{-1}(D) \).

NOTE: here iff = if and only if (a useful shorthand)

Problem 4. Suppose that \( f : A \to B \) and \( g : B \to C \) are functions such that the composite \( g \circ f \) is surjective. Is \( g \) necessarily surjective? What about \( f \)? Give brief proofs or counterexamples.

\( g \) must be surjective. Proof: since \( g \circ f \) is surjective, for every \( c \in C \) there is \( a \in A \) such that \( g \circ f(a) = c \). But \( g \circ f(a) = g(f(a)) \). Hence there is an element \( b \in B \) such that \( g(b) = c \), namely \( b = f(a) \).

But \( f \) need not be surjective because \( g \) need not be injective. eg if \( f : [0, \infty) \to \mathbb{R} \) is \( f(x) = x \), \( f \) is not surjective. Define \( g : \mathbb{R} \to [0, \infty) \) by \( g(y) = y^2 \). Then \( g \circ f : [0, \infty) \to [0, \infty) \) is surjective.

Problem 5. Prove by mathematical induction: \( 3^{2n} - 1 \) is divisible by 8 for all \( n \geq 1 \).

The statement \( P(n) \) is: \( 8 \) divides the integer \( 3^{2n} - 1 \).

Base case: if \( n = 1 \) then \( P(1) \) says that \( 8 \) divides \( 3^2 - 1 = 8 \), which is true.

Inductive step: suppose that \( 8 \) divides \( 3^{2k} - 1 \). We must show that \( 8 \) divides \( 3^{2k+2} - 1 \). But \( 3^{2k+2} - 1 = 3^2 \times 3^{2k} - 1 = 9(3^{2k} - 1) + 9 - 1 = 9(3^{2k} - 1) + 8 \). Since \( 8 \) divides \( 3^{2k} - 1 \) by the inductive hypothesis, it divides \( 9(3^{2k} - 1) \). It also divides \( 8 \). Hence it divides \( 9(3^{2k} - 1) + 8 = 3^{2k+2} - 1 \), as required.