Math 319 Review sheet for Second Midterm

This exam will have 5 questions each worth 10 points. The page of the definitions and theorems now posted will be attached to the exam. One question will ask you to prove a part of one of these theorems. You will have a choice here (see Q1 below.) I have tried to make the other questions very straightforward, like the easier homework problems.

Prove ONE of the following results: (i) Prove that a monotonic increasing sequence that is bounded above converges.

OR: (ii) Suppose \((y_n)\) is a sequence in \(\mathbb{R}\) such that \(\lim y_n = 0\) and suppose that \(|x_n - L| \leq 3|y_n|\) for all \(n \geq 10\). Then \(\lim x_n = L\).

2: Problem 2. Let \(A = \left\{3 + \frac{1}{n} : n \geq 1\right\}\). Which points in \(\mathbb{R}\) are cluster points of \(A\)? Prove all your claims from the definitions.

3: Suppose that \((x_n)\) is a sequence such that the subsequence \((x_{2n})\) converges to 1 and the subsequence \((x_{2n+1})\) converges to 3. Show (from the definition of limit) that \((x_n)\) is not convergent. (Do NOT just quote Thm 3.4.5.)

4: Which of the following sequences are monotonic? Which are convergent? Justify your answers.
   \[
   \begin{align*}
   (i) & \quad x_n = (-1)^n \cos(n\pi); \\
   (ii) & \quad x_n = \frac{n^2}{n + 1}; \\
   (iii) & \quad x_n = \frac{\sin n}{n}.
   \end{align*}
   \]

5: (i) Give an example of a countably infinite subset of \(\mathbb{R}\) that has precisely one cluster point.
   (ii) Adapt your example in (i) so that the set has exactly two cluster points.
   (iii) Give an example of two nonmonotonic sequences \((x_n), (y_n)\) with positive terms whose product is monotonic.
   (iv) Give an example of a sequence that contains a nonconstant monotonic increasing subsequence and a nonconstant monotonic decreasing subsequence. Is there such a sequence that also converges?

Note: I haven’t put any questions exactly like Q5 on the test since they are somewhat nonroutine and therefore hard to do in an exam. But I think this question is good practice for review.

6: Suppose that \((x_n)\) is a convergent sequence with limit \(L\) and that \(x_n \in [0, 2]\) for all \(n\). Prove from the definitions that \(L \in [0, 2]\).

7: Let \(f : (0, 5) \to \mathbb{R}\) be the function \(f(x) = 1/x^2\). Prove from the definition that \(\lim_{x \to 2} f = \frac{1}{4}\).