Math 319 Review sheet for Second Midterm

This exam will have 5 questions each worth 10 points. The page of the definitions and theorems now posted will be attached to the exam. One question will ask you to prove a part of one of these theorems. You will have a choice here (see Q1 below.) I have tried to make the other questions very straightforward, like the easier homework problems.

Prove ONE of the following results: **(i)** Prove that a monotonic increasing sequence that is bounded above converges.
OR: **(ii)** Suppose (y_n) is a sequence in $\mathbb{R}$ such that $\lim y_n = 0$ and suppose that $|x_n - L| \leq 3|y_n|$ for all $n \geq 10$. Then $\lim x_n = L$.

2: Problem 2. Let $A = \left\{3 + \frac{1}{n} : n \geq 1\right\}$. Which points in $\mathbb{R}$ are cluster points of A? Prove all your claims from the definitions.

3: Suppose that (x_n) is a sequence such that the subsequence (x_{2n}) converges to 1 and the subsequence (x_{2n+1}) converges to 3. Show (from the definition of limit) that (x_n) is not convergent. (Do NOT just quote Thm 3.4.5.)

4: Which of the following sequences are monotonic? Which are convergent? Justify your answers.

$$\textbf{(i)} \quad x_n = (-1)^n \cos(n\pi); \qquad \textbf{(ii)} \quad x_n = \frac{n^2}{n+1}; \qquad \textbf{(iii)} \quad x_n = \frac{\sin n}{n}.$$

5: (i) Give an example of a countably infinite subset of $\mathbb{R}$ that has precisely one cluster point.
(ii) Adapt your example in (i) so that the set has exactly two cluster points.
(iii) Give an example of two nonmonotonic sequences $(x_n), (y_n)$ with positive terms whose product is monotonic.

(iv) Give an example of a sequence that contains a nonconstant monotonic increasing subsequence and a nonconstant monotonic decreasing subsequence. Is there such a sequence that also converges?
Note: I haven't put any questions exactly like Q5 on the test since they are somewhat nonroutine and therefore hard to do in an exam. But I think this question is good practice for review.

6: Suppose that (x_n) is a convergent sequence with limit L and that $x_n \in [0, 2]$ for all n. Prove from the definitions that $L \in [0, 2]$.

7: Let $f : (0, 5) \to \mathbb{R}$ be the function $f(x) = 1/x^2$. Prove from the definition that $\lim_{x\to 2} f = \frac{1}{4}$.