Math 319 Review sheet for Final, Dec 2005

This exam will be out of 90 points: each question will be worth 15 points. The exam will be much like Midterm II, EXCEPT that I will ask you to give ONE definition (one of 3.1.3, 3.4.1, 4.1.1, 4.1.4 and 5.1.1) – so the definitions WILL NOT be on the sheet. There still will be a sheet of statements of theorems. These will be much the same as those on last year's exam, but I may update that (If I do, I will give you some warning.)

As in Midterm II, one question will ask you to prove a part of one of the theorems on the sheet. There will be no (hard) questions on the final about countable/uncountable sets or using the Well-ordering principle for $\mathbb{N}$. (But I might ask about sups and infs.)
NOTE: I have written this sheet in a hurry and may revise it SLIGHTLY at the weekend.

Problem 1. (i) Consider the statement:

$$2 + 5 + \cdots + 3n - 1 = \frac{n(3n+1)}{2}.$$

What does it say when $n = 1, n = 2$, and $n = 3$?
(ii) Use mathematical induction to prove it for all n.

Problem 2. Use mathematical induction to show that $n^3 + 3n^2 - n$ is divisible by 3 for all $n \geq 1$.

Problem 3. Prove ONE of the following results:
EITHER: Let $A = (0, 2)$ and suppose that c is a cluster point of the complement $\mathbb{R} \setminus A$. Prove that $c \notin A$.

OR: **(ii)** Let $f : A \to \mathbb{R}$ be any function and let $s := \sup\{f(x) : x \in A\}$. Show that there is a sequence $x_n \in A$ such that $\lim f(x_n) = s$.

Problem 4. (i) Let $x > 0$. Prove that $\lim_{n\to\infty} \frac{1}{1+nx} = 0$.
(ii) Let $0 < b < 1$. Prove that $\lim b^n = 0$.

(Use Archimedes Principle and Bernoulli's inequality, both of which are now on the sheet)

Problem 5. Suppose that $(x_n), (y_n)$ are sequences such that $\lim x_n = 2$ and $\lim y_n = 3$. Prove from the definitions that there is an integer K such that $x_n < y_n$ for all $n \geq K$.

Problem 6. Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = \frac{3x}{x^2+2}$. Prove from the definition that $\lim_{x\to 1} f(x) = 1$.

Problem 7. Describe examples satisfying the following conditions. Justify your answers.
(i) A bounded set A and a continuous function $f : A \to \mathbb{R}$ that is not bounded.

(ii) A convergent sequence that is not monotonic.

(iii) A countable set that is bounded above but does NOT contain its supremum.

Problem 8. Define the sequence (x_n) recursively by setting $x_1 = 1$ and $x_n = 1 + \frac{x_{n-1}}{2}$.
(i) Show that (x_n) is monotonic increasing.
(ii) Show that it converges and find its limit.

Problem 9. Which of the following sequences are convergent? Justify your answers.

$$\textbf{(i)} \quad x_n = \frac{(-1)^n \sin n}{n+1}; \qquad \textbf{(ii)} \quad x_n = \frac{(-2)^n}{n+1}; \qquad \textbf{(iii)} \quad x_n = \frac{n-3}{2n-1}.$$