Answer all the following questions, justifying all your statements. Each question is worth 15 points. There are six questions. Good luck!

1: Prove ONE of the following results:

EITHER: Let $c$ be a cluster point of the set \{ $x_n : n \geq 1$ \}. Show that there is a subsequence of $(x_n)$ that converges to $c$.

OR: Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be continuous and suppose that $x_n$ are points in $\mathbb{R}$ such that $f(x_n) = n$. Prove that the sequence $(x_n)$ is unbounded.
2: Prove from the definition of limit and the results on the sheet that \( \lim_{n \to \infty} \frac{1}{2^n} = 0. \)

3: Consider the function \( f : [0, 2] \to \mathbb{R} \) given by \( f(x) = \frac{2}{1+x} \). Prove from the definition that \( \lim_{x \to 1} f(x) = 1. \)
4: Describe examples satisfying the following conditions. Justify your answers.

(i) an infinite subset of $\mathbb{R}$ that has no cluster points.

(ii) a bounded sequence of real numbers that does not converge.

(iii) a function $f : [-1, 1] \to \mathbb{R}$ that is not continuous at $x = 0$.

5: Let $(x_n)$ be a monotonic decreasing sequence and set $B = \{x_n : n \geq 1\}$. Show that the point $x_5$ is not a cluster point of $B$. 
6: Which of the following sequences are monotonic? Which are convergent?

(i) \( x_n = \frac{n + 1}{2n - 1} \);  
(ii) \( x_n = (-1)^n \frac{n + 1}{2n - 1} \);  
(iii) \( x_n = \frac{n^2 + 1}{2n - 1} \).
Def 3.1.3 A sequence $X = (x_n)$ in $\mathbb{R}$ is said to converge to $x \in \mathbb{R}$ if for every $\epsilon > 0$ there is $K(\epsilon) \in \mathbb{N}$ such that for all $n \geq K(\epsilon)$ the terms $x_n$ satisfy $|x_n - x| < \epsilon$. A sequence that does not converge is called divergent.

Def 3.4.1 Let $X = (x_n)$ be a sequence and $n_1 < n_2 < \cdots < n_k < \ldots$ be a strictly increasing sequence of positive integers. Then the sequence $X' := (x_{n_k})$ given by $(x_{n_1}, x_{n_2}, \ldots)$ is called a subsequence of $X$.

Def 4.1.1. Let $A \subset \mathbb{R}$. A point $c \in \mathbb{R}$ is called a cluster point of $A$ if for every $\delta > 0$ there is at least one point $x \in A$, $x \neq c$ such that $|x - c| < \delta$.

Let $A$ be a subset of $\mathbb{R}$. A point $c \in \mathbb{R}$ is called a boundary point of $A$ if every $\epsilon$-neighborhood of $c$ contains a point of $A$ and a point of its complement $\mathbb{R} \setminus A$. (c does NOT have to be in $A$.) A point $c \in A$ is called an interior point of $A$ if there is $\epsilon > 0$ such that the $\epsilon$-neighborhood of $x$ is entirely contained in $A$.

The subset $A$ of $\mathbb{R}$ is said to be open if for every $x \in A$ there is $\epsilon > 0$ such that the $\epsilon$-neighborhood of $x$ is entirely contained in $A$. The subset $B$ of $\mathbb{R}$ is said to be closed iff its complement $\mathbb{R} \setminus B$ is open.

Def 4.1.4. Let $A \subset \mathbb{R}$ and let $c$ be a cluster point of $A$. A function $f : A \to \mathbb{R}$ is said to have limit $L$ at $c$ if for all $\epsilon > 0$ there is $\delta > 0$ such that $0 < |x - c| < \delta$, $x \in A \implies |f(x) - L| < \epsilon$.

Def 5.1.1. Let $A \subset \mathbb{R}$, let $f : A \to \mathbb{R}$ and let $c \in A$. Then $f$ is continuous at $c$ if for every $\epsilon > 0$ there is $\delta > 0$ such that $|x - c| < \delta$, $x \in A \implies |f(x) - f(c)| < \epsilon$. If $B$ is a subset of $A$ we say that $f$ is continuous on $B$ if it is continuous at all points $b \in B$.

Archimedes Principle: For all $x \in \mathbb{R}$ there is an integer $n > x$.

Bernoulli inequality: For all $x \geq 0$ and $n \geq 1 (1 + x)^n \geq 1 + nx$.

Thm 3.1.10 Comparison theorem for limits. Let $(x_n)$ be a sequence in $\mathbb{R}$ and let $x \in \mathbb{R}$. If $(a_n)$ is a sequence of positive numbers with $\lim a_n = 0$ and if for some $C > 0$ and some $m \in \mathbb{N}$ we have $|x_n - x| \leq Ca_n$ for all $n \geq m$, then $\lim x_n = x$.

Thm 3.2.2 A convergent sequence of real numbers is bounded.

Thm 3.3.2 Monotone Convergence Theorem. A monotone sequence of real numbers is convergent if and only if it is bounded.

Thm 3.4.2 If $X = (x_n)$ converges to $x \in \mathbb{R}$, every subsequence $X'$ of $X$ converges to $x$.

3.4.7: Monotone subsequence theorem. Every sequence has a monotone subsequence.

3.4.8: Bolzano–Weierstrass theorem. A bounded sequence of real numbers has a convergent subsequence.

Thm 4.1.8. Sequential criterion. Let $f : A \to \mathbb{R}$ and $c$ be a cluster point of $A$. Then $\lim_{x \to c} f = L$ iff for every $(x_n)$ in $A \setminus \{c\}$ with limit $c$ the sequence $(f(x_n))$ converges to $L$.

Thm 5.1.3. Sequential criterion for continuity: $f : A \to \mathbb{R}$ is continuous at $c \in A$ iff for every $(x_n)$ in $A$ that converges to $c$ the sequence $(f(x_n))$ converges to $f(c)$.

Thm 5.3.2 Let $I = [a, b]$ be a closed bounded interval and $f : I \to \mathbb{R}$ be continuous on $I$. Then $f$ is bounded on $I$, i.e. there is $M$ such that $|f(x)| \leq M$ for all $x \in I$.

Thm 5.3.4 Let $I = [a, b]$ be a closed bounded interval and $f : I \to \mathbb{R}$ be continuous on $I$. Then $f$ has an absolute maximum and an absolute minimum on $I$, i.e. there are points $c, d$ in $I$ such that $f(c) \leq f(x) \leq f(d)$ for all $x \in I$. 