Project 3

(due by 12/14/06 - 5:00pm)

This project concerns some simple applications of Linear Algebra to Fibonacci Numbers. Discuss the problem below in a concise and precise essay at most 5 typed pages long. Whenever you use a reference, quote it and do not copy. Use your own words.

Let $a_n$ denote the basic sequence of Fibonacci numbers defined by the recursive relation

$$a_{n+2} = a_n + a_{n+1}, \quad a_0 = 1, a_1 = 1.$$ 

Work out the problems below and find an explicit formula for $a_n$.

1. Consider an operator $T$ on the vector space $\mathbb{R}^2$ such that $T$ maps the vector $(x, y)$ to the vector $(y, x + y)$. Show that $T$ maps $(a_{n-2}, a_{n-1})$ to $(a_{n-1}, a_n)$, where $a_n$ is the Fibonacci sequence. Write the matrix $A$ of $T$ in the standard basis and prove that

$$A^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix}.$$

2. Diagonalize the operator $T$ by finding its eigenvalues and eigenvectors. Show that the eigenvectors $v_1, v_2$ form a basis in $\mathbb{R}^2$. If $B$ denotes the diagonal matrix of $T$ with respect to this basis, verify that $A = P^{-1}BP$, where the columns of the transition matrix $P$ are $v_1, v_2$ in terms of the coordinates with respect to the standard basis.

3. Find $B^n$ and conclude that $A^n = P^{-1}B^nP$ from $A = P^{-1}BP$. Now easily find $A^n$ and write an explicit formula for the $n$-th Fibonacci number.

4. Suppose that the numbers $b_n, n \geq 0$, are defined by the same recursive relation, but with $b_0 = 1, b_2 = 3$. Thus $b_2 = 4, b_3 = 7$. . . . Find an explicit formula for $b_n$. Check your answer for $b_{10}$ (using a calculator).