Math 310: Midterm 2 (slightly edited)
November 16, 2006

Name: ID number:

There are 4 questions worth a total of 100 points, plus one small bonus question worth 10 points. Please justify all your statements, and write neatly so that we can read and follow your answers. Any theorems that you use in your arguments should be carefully stated. Continue your answers on the back of the pages. Also, please turn off cell phones.

Question 1. (30 points) Let \((V, \langle \cdot, \cdot \rangle)\) be a finite dimensional inner product space over \(\mathbb{R}\).

(i) Define the length \(\|v\|\) of a vector \(v \in V\).

(ii) Show from this definition that \(\|v\|^2 + \|w\|^2 = \|v + w\|^2\) if and only if the vectors \(v, w \in V\) are orthogonal.

(iii) Find an orthonormal basis for the subspace \(x_1 + 2x_2 - x_3 = 0\) of \(\mathbb{R}^3\).

(iv) Find the orthogonal projection of \(y = (1, 1, 1)\) onto this subspace.

Notes: When proving (ii) work with the inner product \(\langle v, w \rangle\). The subspace \(U\) in (iii) is a plane, so the basis should have two vectors in it. (Many of you gave me one vector, a multiple of \((1, 2, -1)\), i.e. a basis for \(U^\perp\).)
Question 2: (20 points) Let $V$ be a finite dimensional vector space over $\mathbb{C}$ and suppose that $S, T \in \mathcal{L}(V)$ commute.

(i) Show that null $S$ and range $S$ are invariant under $T$.

(ii) Suppose in addition that $V = \text{null } S \oplus \text{range } S$ where both null $S$ and range $S$ have nonzero dimension. Show that $T$ has at least two linearly independent eigenvectors.

Note: (ii) is an easy deduction from one of the theorems in the book; you should say which one and explain what is going on.
Question 3: (25 points) (i) Let \( A = \begin{bmatrix} 0 & -1 \\ 4 & 4 \end{bmatrix} \). Find a basis of \( \mathbb{C}^2 \) such that the operator \( T_A \) defined by \( T_Av = Av \) is represented by an upper triangular matrix \( M \) with respect to this basis. What is \( M \)?

(ii) (10 points). Suppose that \( A \) is a 3 \( \times \) 3 matrix of the form \( \begin{bmatrix} 1 & * & * \\ 0 & -1 & * \\ 0 & 0 & 4 \end{bmatrix} \), where the entries * are all nonzero. Is there always a basis of \( \mathbb{C}^3 \) with respect to which \( T_A \in \mathcal{L}(\mathbb{C}^3) \) can be represented by a diagonal matrix?

Note: for (i) you should begin by finding the eigenvalues and eigenvectors of \( A \). It turns out that these do not form a basis for \( \mathbb{C}^2 \), so you have to figure out what to do to complete the proof. (ii) is an easy deduction from some theorems.
Question 4: (15 points) (i) Let $(V, \langle , \rangle)$ be a finite dimensional inner product space, and let $T \in \mathcal{L}(V)$. Define the adjoint $T^*$ of $T$.

(ii) Show that if the subspace $U$ of $V$ is invariant under $T$ then $U^\perp$ is invariant under $T^*$.

(iii) (Bonus) (10 points) Give an example of an operator $T \in \mathcal{L}(\mathbb{C}^2)$ that has a 1-dimensional invariant subspace $U$ such that $U^\perp$ is NOT invariant under $T$.

Note: in (ii) you should work from the definition you gave in (i).