Math 310: Homework 8
due Nov 8, 9 2006 in recitation

Ex 1 Let $U \subset \mathbb{R}^4$ be the subspace given by the equations $x_1 + x_2 + x_3 + x_4 = 0$, $x_1 - x_2 + 2x_3 + x_4 = 0$.

(i) Find a basis of $U$. (Make the calculations easier by giving the vectors lots of zeros...)

(ii) Find an orthonormal basis of $U$.

(iii) Extend this to an orthonormal basis for $\mathbb{R}^4$. (First find any extension and then apply Gram–Schmidt.)

(iv) Let $v = (1, 2, 3, 4)$. Find the coordinates of $V$ with respect to the basis you found in (iii).

Ex 2. Let $e_1, \ldots, e_n$ be any basis of an inner product space $V$. Define $U := \text{span}(e_1, \ldots, e_k)$ and $W = \text{span}(e_{k+1}, \ldots, e_n)$. Also define $U^\perp := \{v : \langle v, u \rangle = 0, \forall u \in U\}$. ($U^\perp$ is called the orthogonal complement to $U$.)

(i) Show that $V = U \oplus W$.

(ii) Show that $U^\perp$ is a subspace. Show also that $U^\perp = W$.

(iii) Deduce that for any subspace $U$ of $V$, $V = U \oplus U^\perp$.

(iv) Now assume that $V = \mathbb{R}^4$ and that $U$ is the subspace defined in Ex 1. Calculate the decomposition of $v = (1, 1, 1, 1)$ as a sum $u + w$ where $u \in U$ and $w \in U^\perp$.

Ex 3 Let $V$ be the vector space of all $n \times n$ matrices over $\mathbb{R}$, and given any two matrices $A, B \in V$ define

$$\langle A, B \rangle = \text{trace}(AB) = \sum_{i,j} a_{ij}b_{ji}.$$ 

(i) Show that this satisfies all axioms for an inner product except possibly for positivity and nondegeneracy. (e.g. give an example (with $n = 2$) such that $A \neq 0$ but $\text{trace}A^2 = 0$.)

(ii) If $A$ is a real symmetric matrix, show that $\text{trace}(A^2) \geq 0$, and $\text{trace}(A^2) > 0$ if $A \neq 0$. Thus the trace defines an inner product on the space of real symmetric matrices.

(iii) Let $V$ be the symmetric space of real $n \times n$ symmetric matrices. What is dim $V$? What is the dimension of the subspace $W$ consisting of those matrices $A$ such that $\text{trace}(A) = 0$? What is the dimension of the orthogonal complement $W^\perp$ relative to the inner product defined above?

Ex 4 Let $A$ be an $n \times n$ matrix, and define $T \in \mathcal{L}(\mathbb{R}^n)$ by $Tv = Av$.

(i) Show that $T$ is diagonalizable iff there exists an invertible matrix $Q$ such that $Q^{-1}AQ$ is a diagonal matrix.

(ii) How can you interpret the columns of the matrix $Q$? (Hint: think of these as vectors. What relation do they have to the operator $T$?)
**Ex 5** Two linear operators $S$ and $T$ on a finite-dimensional vector space $V$ are called *simultaneously diagonalizable* if there exists a basis $\mathcal{B}$ for $V$ such that both $M(S, \mathcal{B})$ and $M(T, \mathcal{B})$ are diagonal matrices. This is equivalent to saying that there is a basis for $V$ consisting of vectors that are eigenvectors for both $S$ and $T$.

(i) Prove that if $S$ and $T$ are simultaneously diagonalizable operators then $S$ and $T$ commute. (Hint: see what the operators $ST$ and $TS$ do to a suitable basis for $V$.)

(ii) (Bonus) Prove also that if $S$ and $T$ are diagonalizable operators that commute then they are simultaneously diagonalizable.

(iii) Let $T_A, T_B \in \mathcal{L}(\mathbb{F}^n)$ be the operators defined by multiplication by the matrices $A, B$. Show that $T_A, T_B$ are simultaneously diagonalizable iff there is an invertible matrix $Q$ such that both $Q^{-1}AQ$ and $Q^{-1}BQ$ are diagonal matrices. (cf Ex 4).

(iv) (Bonus) Deduce that if the matrices $A, B$ commute there is an invertible matrix $Q$ such that both $Q^{-1}AQ$ and $Q^{-1}BQ$ are diagonal matrices.