Ex 1 Let \( T_1, T_2, T_3 \in \mathcal{L}(\mathbb{C}^3) \) be the linear maps given by the following matrices:

\[
A_1 := \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad A_2 := \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad A_3 := \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.
\]

Find their minimal and characteristic polynomials.

(ii) Let \( T \in \mathcal{L}(V) \) where dim \( V = n \). Suppose that \( q(t) \) is a monic polynomial of degree \( n \) such that \( q(T) = 0 \). What conditions guarantee that \( q(T) \) is the characteristic polynomial of \( T \)? Discuss this question using the maps in (i) as examples.

Ex 2 (i) Let dim \( V = n \). Suppose that for some vector \( v \in V \) the list \( v_0 := v, v_1 := Tv, \ldots, v_{n-1} := T^{n-1}v \) is linearly independent. Why is there a linear relation

\[ T^n v = a_0 v_0 + a_1 v_1 + \cdots + a_{n-1} v_{n-1} \]

(ii) Let \( q(t) = t^n - a_{n-1} t^{n-1} - \cdots - a_1 t - a_0 \). Show that \( q(T) v_i = 0 \) for all \( i \). (Use the fact that \( v_i = T^i(v_0) \) for all \( i \).) Hence deduce that \( q(T) = 0 \).

(iii) Show also that there is no polynomial \( m \) of degree < \( n \) such that \( m(T) = 0 \). Hence deduce that \( q(t) \) is the minimal and the characteristic polynomial of \( T \). (Compare Ex 1.)

(iv) Use this method to find the characteristic polynomial of \( T_A : \mathbb{C}^3 \to \mathbb{C}^3 \) where

\[
A := \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}, \quad \text{and} \quad v = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.
\]

(v) What are the eigenvalues and eigenvectors of \( T_A \)? (You can use any method here.)

Ex 3 (i) Consider the permutations on 8 letters given by the arrays

\[
\mathbf{m} := [m_1, \ldots, m_8] = [3, 5, 1, 6, 7, 2, 8, 4], \quad \mathbf{n} := [n_1, \ldots, n_8] = [3, 8, 1, 6, 7, 2, 5, 4].
\]

Calculate \( \text{sign} \mathbf{m} \) and \( \text{sign} \mathbf{n} \). Why do you expect \( \text{sign} \mathbf{m} = -\text{sign} \mathbf{n} \)?

(ii) A permutation \( \mathbf{m} \) is a map \( \{1, \ldots, 8\} \to \{1, \ldots, 8\} \) given by \( m(1) = m_1, \ldots, m(8) = m_8 \). To emphasize this one can describe \( \mathbf{m} \) by two rows:

\[
\mathbf{m} := \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 5 & 1 & 6 & 7 & 2 & 8 & 4 \end{bmatrix}.
\]

Another way of describing a permutation is in terms of cycles \( (d_1, d_2, \ldots, d_r) \): this is the map that takes \( d_1 \) to \( d_2 \), \( d_2 \) to \( d_3 \) and so on, finally taking \( d_r \) back to \( d_1 \). In this notation, when one writes a product one does the right hand one first. Thus:

\[
(134)(24) = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{bmatrix}.
\]

Note that \( (134)(24) = (1342) \).

(a) Write down \( (15)(245)(34) \) in the double row format.
(b) Hence write \((15)(245)(34)\) as a product of disjoint cycles (ie so that no number occurs in more than one cycle.)

(c) Repeat the above two steps with \((23)(245)(34)\)

(d) Write down the permutation \(m\) above as a product of disjoint cycles.

(iii) Use (a) (b) above to write down \((15)(245)(34)\) as a product of transpositions in two different ways. Check that the number of transpositions has the same parity in both cases. (A transposition is a cycle of length 2. We saw in class that any permutation can be written as a product of transpositions. For example \((1234) = (43)(42)(41)\).

(iv) Use (d) to write \(m\) as a product of \(k\) transpositions. Check that \((-1)^k = \text{sign } m\).

**Ex 4** Let

\[
B = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
2 & 0 & 1 & 0 & 4 \\
0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 2 \\
3 & 0 & 1 & 0 & 0 \\
\end{bmatrix}.
\]

Recall that

\[
\det B = \sum_{m \in \text{perm}_5} \text{sign } m \ b_{m_1,1} b_{m_2,2} b_{m_3,3} b_{m_4,4} b_{m_5,5}.
\]

Since \(m\) is a permutation, each term in this sum contains just one element from each row and each column of \(B\). For \(B\) as above, which permutations give you nontrivial terms in this sum? (Hint: since for example there is just one element in the first row which lies in position \(b_{12}\) we must have \(m_2 = 1\).) List all these permutations, and hence calculate \(\det B\).

**Ex 5** (i) The trace \(\text{tr} A\) of a matrix \(A\) is the sum of its diagonal entries. Explain why \(\text{tr} A\) is the sum of the eigenvalues of the linear map \(T_A\).

(ii) Suppose that \(A\) is a complex \(n \times n\) matrix such that \(A^2 = A\). Show that \(\text{tr} A\) is a nonnegative integer. (Hint: What can you say about the eigenvalues of \(A\)?)

**Ex 6** (Bonus) Suppose that in the situation of Ex 1 the span of the \(T^i v, i = 0, \ldots, n - 1\), has dimension \(< n\). You can still find a polynomial \(f(t)\) such that \(f(T)(T^i v) = 0\) for all \(i\). What can you say about its roots? What relation does this have to the minimal or characteristic polynomial? You could experiment starting with the matrix

\[
A := \begin{bmatrix}
1 & 1 & 0 \\
0 & 2 & 0 \\
1 & 0 & 3 \\
\end{bmatrix}, \text{ and } v = \begin{bmatrix}
1 \\
0 \\
0 \\
\end{bmatrix}.
\]