HW12 Solutions

Sec 6.2 :

Prob. 2: The matrix can be easily transformed into the following form using Gaussian elimination,
\[
\begin{bmatrix}
1 & 2 & 3 \\
0 & 4 & 5 \\
0 & 0 & 6 \\
\end{bmatrix}
\]
and its determinant equals 1.4.6=24. *No row swap or multiplication was used*

Prob. 10: After successive elimination steps, the matrix becomes,
\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
0 & 1 & 2 & 3 & 4 \\
0 & 0 & 1 & 3 & 6 \\
0 & 0 & 0 & 1 & 4 \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]
So, the determinant equals unity. *No row swap or multiplication was used*

Prob. 12:
\[
\begin{bmatrix}
V_4 \\
V_2 \\
V_3 \\
V_1 \\
\end{bmatrix}
\rightarrow \quad \text{one swap} \quad \begin{bmatrix}
V_1 \\
V_2 \\
V_3 \\
V_4 \\
\end{bmatrix}
\]
So, \( \text{det} = 8 \cdot (-1) = -8 \)

Prob. 14:
\[
\begin{bmatrix}
V_1 \\
V_2 + 9V_4 \\
V_3 \\
V_4 \\
\end{bmatrix}
\rightarrow \quad \begin{bmatrix}
V_1 \\
V_2 \\
V_3 \\
V_4 \\
\end{bmatrix}
\]
So, \( \text{det 1}^{\text{st}} = \text{det 2}^{\text{nd}} \)

Prob. 30:
\[F(t) = \text{det} \begin{bmatrix}
1 & 1 & 1 \\
a & b & t \\
a^2 & b^2 & t^2 \\
\end{bmatrix}\]
To prove that the function is quadratic, enough to show that the coefficient of \( t^2 \) is not zero. If we used the third column to evaluate the matrix, we would find that the coefficient of \( t^2 \) is \((b-a)\). So, it’s quadratic unless \((b-a)\) vanishes.

\( \text{b) If } t = a \text{ or } t = b, \text{ we’ll have two identical columns. What is the value of this determinant?} \)

Let’s look at the matrix transpose.
\[
\begin{bmatrix}
1 & a & a^2 \\
1 & b & b^2 \\
1 & a & a^2 \\
\end{bmatrix}
\]
It will be something like, \(1^{\text{st}} \) row is the same like \(3^{\text{rd}} \). So, the system of equations is clearly inconsistent. The determinant is necessarily zero. Using fact 6.2.7, we prove that the \( f(a) = f(b) = 0 \)

Another way to show that, is to think of the matrix that has two identical columns as if it expresses a volume in 3 dimensions. If two edges of that volume are expressed with the same vector, this means that it is a shape in two dimensions instead of three; the volume = zero. \( \rightarrow \) determinant is zero.
If \( f(t) = k(t-a)(t-b) \), \( k \) is the coefficient of \( t^2 \). From part (a), \( k = (b-a) \)

c) \( f(t) = k(t-a)(t-b) \). Obviously, the matrix will be **noninvertible** when

\[
\text{Det} = f(t) = 0 \Rightarrow t = a \text{ or } t = b
\]

**Prob. 38:**

\[
\text{Det} (A^T \cdot A) = \text{Det} A \cdot \text{Det} A = (\text{Det} A)^2 = 3^2 = 9
\]

**Sec 6.3 :**

**Prob. 2:**

Form a matrix out of the two vectors as \[
\begin{bmatrix}
3 & 8 \\
7 & 2 \\
\end{bmatrix}
\]. The determinant of this matrix is twice the area surrounded by the two vectors. Area = \( \text{Det}/2 = \frac{|6-56|}{2} = 25 \)

**Prob. 7:**

Form a matrix out of each two successive pair of vectors. Four matrices are produced. Their determinants give twice of the area shaded in the graph

\[
\text{Area} = \frac{1}{2} \left( \text{Det} \begin{bmatrix}
5 & -7 \\
5 & 7 \\
\end{bmatrix} + \text{Det} \begin{bmatrix}
-7 & -5 \\
7 & -6 \\
\end{bmatrix} + \text{Det} \begin{bmatrix}
-5 & 3 \\
-6 & -4 \\
\end{bmatrix} + \text{Det} \begin{bmatrix}
3 & 5 \\
-4 & 5 \\
\end{bmatrix} \right) = 110
\]

**Prob. 24:**

Using Cramer’s rule,

\[
X = \begin{bmatrix}
8 & 3 & 0 \\
3 & 4 & 5 \\
-1 & 0 & 7 \\
2 & 3 & 0 \\
0 & 4 & 5 \\
6 & 0 & 7 \\
\end{bmatrix} = 1 \quad Y = \begin{bmatrix}
2 & 8 & 0 \\
0 & 3 & 5 \\
6 & -1 & 7 \\
2 & 3 & 0 \\
0 & 4 & 5 \\
6 & 0 & 7 \\
\end{bmatrix} = 2 \quad Z = \begin{bmatrix}
2 & 3 & 8 \\
0 & 4 & 3 \\
6 & 0 & -1 \\
2 & 3 & 0 \\
0 & 4 & 5 \\
6 & 0 & 7 \\
\end{bmatrix} = -1
\]

Check by yourself that these values satisfy the equation system.

**Prob. 24:**

The classical adjoint should NOT be calculated from the inverse matrix. Actually, most of the time, we calculate the adjoint in order to obtain the inverse.

**• How to calculate the classical adjoint :**

1) Calculate the **matrix of the minor determinants** of each element in the matrix

\[
\begin{bmatrix}
1 & 0 & 0 \\
2 & 3 & 0 \\
4 & 5 & 6 \\
\end{bmatrix}
\]

i.e.) from the matrix, \[
\begin{bmatrix}
18 & 12 & -2 \\
0 & 6 & 5 \\
0 & 0 & 3 \\
\end{bmatrix}
\]

*take a look at “Minors” at page 250*
2) Multiply each element by the sign in its location from 

\[
\begin{bmatrix}
+ & - & + \\
- & + & - \\
+ & - & + \\
\end{bmatrix}
\]

i.e.) our matrix here becomes, 

\[
\begin{bmatrix}
18 & -12 & -2 \\
0 & 6 & -5 \\
0 & 0 & 3 \\
\end{bmatrix}
\]

3) Take the Transpose, 

\[
\begin{bmatrix}
18 & 0 & 0 \\
-12 & 6 & 0 \\
-2 & -5 & 3 \\
\end{bmatrix}
\]

i.e.) and that is the final result; the classical adjoint

Apply this method to any square matrix of any size.

To check that the answer is correct, we know that,

\[A^{-1} = \frac{A^{\text{adjoint}}}{\text{Det}A} \Rightarrow A A^{-1} = I = \frac{A A^{\text{adjoint}}}{\text{Det}A} \Rightarrow A A^{\text{adjoint}} = \text{Det}A I \]

\[I, \text{is the identity matrix}\]

So, you can multiply the adjoint matrix with the one you started with. The result is expected to be a matrix with equal diagonal elements and zero’s elsewhere (the identity matrix multiplied by the determinant value)