This integral vanishes for any \((f(t), g(t)) = \text{odd function}\)\
\[ \Rightarrow \text{if } f(t) = t \Rightarrow g(t) \text{ must contain } t, t^2 \text{ components only} \]
\[ \text{in general, } g(t) = a + bt^2 \]
This is the space, but we need basis. Could it be \(a, bt^2\)? Let's check.

Normalization,
\[ \langle a, a \rangle = \frac{1}{2} \int_{-1}^{1} a \cdot a \, dt = 1 \Rightarrow a^2 = 1 \rightarrow a = \pm 1 \]
\[ \langle bt^2, bt^2 \rangle = \frac{1}{2} \left[ \frac{1}{2} b^2 \cdot (t^2) \right]_{-1}^{1} = \frac{1}{2} b^2 \frac{4}{5} = 1 \]
\[ \Rightarrow \frac{b^2}{5} = 1 \Rightarrow b = \pm \sqrt{5} \]

It seems that \((\pm 1, \pm \sqrt{5} t^2)\) might work as a basis. However, they are not orthogonal to each other. Their product is even function and the integral will not vanish.

There is two ways to find orthonormal basis,

# Method one, start with \(1, t^2\). These are neither...
orthogonal nor parallel. Use these as starting vectors in "Gram-Schmidt" process

\[ u_1 = \frac{\hat{v}_1}{\|\hat{v}_1\|} = 1 \quad \hat{v}_2 = \hat{v}_2 - (u_1, \hat{v}_2) u_1 \]

\[ = \frac{t^2 - \frac{1}{3}}{\sqrt{\frac{4}{145}}} \int_{-1}^{1} (1 - t^2) \, dt = t^2 - \frac{1}{3} \]

\[ \|\hat{v}_2\| = \sqrt{\frac{4}{145}} \]

\[ \Rightarrow u_2 = \frac{t^2 - \frac{1}{3}}{\sqrt{\frac{4}{145}}} \]

\[ \Rightarrow u_1, u_2 \text{ form the orthonormal basis} \]

\[ \text{Method two,} \]

The space is \( a + bt^2 \). Dimension = 2. Assume a very general basis, \( \left\{ a_1 + b_1 t^2 \right\} \)

\( \left\{ a_2 + b_2 t^2 \right\} \)

we need to satisfy three conditions:

1) \( \frac{1}{2} \int_{-1}^{1} (a_1 + b_1 t^2)(a_1 + b_1 t^2) \, dt = 1 \) \( \rightarrow \) normalization

2) \( \frac{1}{2} \int_{-1}^{1} (a_2 + b_2 t^2)(a_2 + b_2 t^2) \, dt = 1 \)

3) \( \frac{1}{2} \int_{-1}^{1} (a_1 + b_1 t^2)(a_2 + b_2 t^2) \, dt = 0 \) \( \rightarrow \) orthogonality
Three equations in Four variables \( \rightarrow \) Many solutions

And this makes sense because the basis choice is not unique.

You may choose \([a_1=1]\) and solve the three eqn system to find an orthonormal basis.

\[ \text{Sec 6.1} \]

8) \( \text{Det} = 1 \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} - 2 \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix} + 3 \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix} = -1 + 4 - 3 = 0 \)

\( \Rightarrow \) Matrix is non-invertible

30) \[ \begin{bmatrix} 4 & 2 & 0 \\ 4 & 6 & 0 \\ 5 & 2 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0 \]

\( \Rightarrow (3-\lambda) \left[ (6-\lambda)(4-\lambda) - 8 \right] = 0 \)

\( \left( \lambda^2 - 10\lambda + 16 \right) = (\lambda - 2)(\lambda - 8) \)

\( \Rightarrow \lambda = 2, 3, 8 \)

34) \( \text{Det} = -4 \begin{bmatrix} 4 & 5 & 0 \\ 3 & 6 & 0 \\ 1 & 8 & 2 \end{bmatrix} + 3 \begin{bmatrix} 4 & 5 & 0 \\ 3 & 6 & 0 \\ 2 & 7 & 1 \end{bmatrix} \)

\( = 2(24 - 15) - 1(24 - 15) \)

\( = -8 \times 9 + 3 \times 9 = -45 \)