A General Notion of Shears, and Applications

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1. INTRODUCTION

In this paper we introduce a generalization of the notions of shears and overshears to arbitrary complex manifolds. The concept is very simple, but is far-reaching in the study of complex manifolds having very large automorphism groups. We explore below some of the consequences of this new concept in connection with the density property, which we now recall.

In [V1] we introduced the notion of complex manifolds with the *density property*. Recall that a complex manifold M has the density property if the Lie subalgebra of $\mathcal{X}_{\mathcal{O}}(M)$ generated by the complete vector fields on M is a dense subalgebra. More generally, a Lie subalgebra $\mathfrak{g} \subset \mathcal{X}_{\mathcal{O}}(M)$ is said to have the density property if the complete vector fields in $\mathfrak{g}$ generate a dense subalgebra of $\mathfrak{g}$. (So M has the density property if and only if $\mathcal{X}_{\mathcal{O}}(M)$ has the density property.) Another important case occurs when M has a nonvanishing holomorphic n-form ($n = dim_{\mathbb{C}} M$), i.e., a holomorphic volume element, ω. We say that (M, ω) has the *volume density property* if the Lie algebra $\mathcal{X}_{\mathcal{O}}(M, \omega) := \{X \in \mathcal{X}_{\mathcal{O}}(M) \ ; \ L_X \omega = 0\}$ has the density property. Andersén proved that $(\mathbb{C}^n, dz_1 \wedge ... \wedge dz_n)$ has the volume density property [A], and then Andersén and Lempert proved that $\mathbb{C}^n$ has the density property [AL]. The author showed that for every complex Lie group G, $(G \times \mathbb{C}, \omega)$ has the volume density property, where ω is the unique (up to constant multiple) left (or right) invariant holomorphic volume element on $G \times \mathbb{C}$, and that if G is a Stein Lie group, then $G \times \mathbb{C}$ has the density property. The author also produced several examples of Lie algebras of vector fields with the density property.

In [V2]we used jets to explore the complex structure of (mostly Stein) complex manifolds with the density property. It was shown, among other things, that Stein manifolds with the density property admit open subsets biholomorphic to $\mathbb{C}^n$ and have interesting properties with respect to their embedded submanifolds. Some of the results were known for $\mathbb{C}^n$ through works of Buzzard, Fornaess, Forstnerič, Globevnik, Rosay, Stensønes and others.

With the usefulness of the density property already established in the literature, some sort of classification or fine structure theorem is very desirable. Such a result seems at the moment very far off, in part due to the lack of examples. The main theorems of this paper, which we now state, give many new examples of the density property; more importantly, the proofs establish techniques which can be used to construct other examples. We shall pursue this in future work.

Theorem 1 *Let $M^2 := \mathbb{C}^2 \backslash \{xy = 1\}$ and $\omega := (xy-1)^{-1} dx \wedge dy$. Then (M^2, ω) has the volume density property*

The study of the space M^2 was inspired by discussions with Rosay several years ago. M^2 is important because it is another instance of the mysterious pre-phenomenon (we say "pre" because there are no proofs that it exists) of a holomorphic volume element which is preserved by every holomorphic automorphism.

In the next result, we study a complex Lie group which is not of the form $G \times \mathbb{C}$. There is, as of yet, no general theory here, so we focus on one example.

Theorem 2 *The complex Lie group $Sl(2, \mathbb{C}) := \{(a, b, c, d) \in \mathbb{C}^4 \mid ad - bc = 1\}$ has the density and volume density property.*

Next we introduce a new class of complex manifolds with holomorphic volume element called EMV manifolds. These spaces are generalizations of complex Lie groups, but also of certain (e.g. Stein, but also some other) complex homogeneous spaces. Roughly speaking, they have the property that all holomorphic vector fields on them can be approximately written as finite sums of the form $\sum f_j X_j$ where f_j are *any* holomorphic functions, and X_j are divergence zero $\mathbb{C}$-completely generated (see section 2 for the definition) holomorphic vector fields.

Theorem 3 *Let (M, ω) be an EMV manifold. Then $(M \times \mathbb{C}, \omega \wedge dz)$ has the volume density property. If M is moreover an open subset of a Stein manifold, then $M \times \mathbb{C}$ has the density property.*

As already suggested, the key tool used in the proofs of these theorems is a generalization to arbitrary complex manifolds of the notion of *shears* and *overshears.* This tool may have some independent interest as well. The idea is quite simple: given a $\mathbb{C}$-complete holomorphic vector field X in a complex manifold M, one tries to produce new complete vector fields of the form $f \cdot X$, with $f \in \mathcal{O}(M)$. We establish necessary and sufficient conditions on such f, and these conditions define in a natural way function spaces associated to X. We then prove theorems to the effect that the structure of these function spaces depends on the intrinsic and extrinsic geometry of the orbits of X.

The organization of the paper is as follows. In section 2 we briefly recall some basic definitions in the theory of ordinary differential equations and volume geometry, taking the opportunity to establish notation. In section 3 we introduce and develop general shears and overshears. In part, our results here explain why it was easiest to prove the density property for spaces of the form $G \times \mathbb{C}$. In section 4 we prove theorem 1, and in section 5 we prove theorem 2. The proofs are rather combinatorial in nature. In section 6 we introduce EM and EMV spaces, and prove theorem 3 as well as some related results. Finally, in section 7 we state a question which naturally arises in the course of the paper, and give an example of a complex manifold which may or may not have the (volume) density property, but for which the combinatorial methods of sections 4 and 5 become too cumbersome to carry out.

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2. SOME PRELIMINARIES

In this section we recall a few basic concepts and establish the notation used below.

A holomorphic vector field X is a holomorphic section of $T^{1,0}M$, the holomorphic part of the complexified tangent bundle. Since there is a natural identification of $T^{1,0}M$ with the real tangent bundle TM, we can identify X with a real vector field, which we still denote by X. This vector field has a flow φ_X, which is a map defined on an open subset of $M \times \mathbb{R}$ containing $M \times \{0\}$ as follows: for $(x,t) \in M \times \mathbb{R}$, $\varphi_X^t(x)$ is the point $c(t) \in M$, where $c : I \subset \mathbb{R} \to M$ is the maximal solution of the initial value problem

$$\frac{dc}{dt} = X(c), \qquad c(0) = x.$$

Moreover, φ_X^t is holomorphic for each t. We denote the set of holomorphic vector fields on M by $\mathcal{X}_{\mathcal{O}}(M)$.

A holomorphic vector field X is called *complete* if φ_X is defined on all of $M \times \mathbb{R}$. In this case $\{\varphi_X^t \mid t \in \mathbb{R}\}$ is a one-parameter group of automorphisms of M.

X is called $\mathbb{C}$-*complete* if both X and iX are complete. Let $\psi^{s+it}(x) := \varphi_X^s \circ \varphi_{iX}^t(x)$. One checks that, since $[X, iX] = 0$ for all holomorphic vector fields, $\{\psi^\zeta \mid \zeta \in \mathbb{C}\}$ defines a complex one-parameter group of automorphisms which is holomorphic in ζ, i.e., a holomorphic $\mathbb{C}$-action. In this paper we shall use the phrase *complete* to mean $\mathbb{C}$-complete.

$\mathcal{X}_{\mathcal{O}}(M)$ is equipped with a bracket, or commutator, operation $[X,Y] = XY - YX$ which makes it into a Lie algebra. Given any Lie algebra $\mathfrak{g}$ of holomorphic vector fields, we can consider the Lie subalgebra $\mathfrak{g}'$ of $\mathfrak{g}$ generated by the complete vector fields in $\mathfrak{g}$. Any $X \in \overline{\mathfrak{g}'}$ is said to be $\mathfrak{g}$-*completely generated.* If $\mathfrak{g} = \mathcal{X}_{\mathcal{O}}(M)$ we omit reference to the Lie algebra. If $\mathfrak{g} = \mathcal{X}_{\mathcal{O}}(M, \omega)$ (see below) we say that $X \in \overline{\mathfrak{g}'}$ is divergence zero completely generated.

Let us now suppose that M admits a nowhere vanishing holomorphic n-form ω, where $n = dim_{\mathbb{C}} M$. We call such a form a *holomorphic volume element.* Given a holomorphic volume element ω, we can define a map $div_\omega : \mathcal{X}_{\mathcal{O}}(M) \to \mathcal{O}(M)$ by

$$div_\omega(X) = \frac{L_X \omega}{\omega},$$

where L_X is the Lie derivative of X:

$$L_X \alpha = \left.\frac{d}{dt}\right|_{t=0} (\varphi_X^t)^* \alpha.$$

Since $L_{[X,Y]} = L_X L_Y - L_Y L_X$, one easily shows that

$$div_\omega([X,Y]) = X div_\omega Y - Y div_\omega X.$$

Another useful formula, due to H. Cartan, is

$$div_\omega(X) = \frac{d(i_X\omega)}{\omega},$$

where i_X is contraction with respect to X.

Finally, we denote the kernel of div_ω by $\mathcal{X}_{\mathcal{O}}(M,\omega)$, and call $X \in \mathcal{X}_{\mathcal{O}}(M,\omega)$ a *divergence zero* vector field.

3. GENERAL SHEARS AND OVERSHEARS

Basic propositions and the definition. Let $X \in \mathcal{X}_{\mathcal{O}}(M)$. We define

$$I^j(X) = I^j_{\mathcal{O}}(X) := \{ f \in \mathcal{O}(M) \mid X^j f = 0 \}.$$

If $f \in I^1(X)$ (resp. $I^2(X)$) we say f is a first (resp. second) integral of X. The following proposition is immediate.

Proposition 3.1. *Let X be a holomorphic vector field with (local) flow g^t_X. Then $f \in I^1(X)$ (resp. $I^2(X)$) if and only if (where defined)*

$$f \circ g^t_X = f \quad (\textit{resp. } f \circ g^t_X = f + tXf).$$

While first integrals have been studied extensively in the past, second integrals seem not to have been looked at. However, in the holomorphic category it is natural to study first and second integrals because of the following fundamental proposition.

Proposition 3.2. *If $X \in \mathcal{X}_{\mathcal{O}}(M)$ is $\mathbb{C}$-complete and $f \in \mathcal{O}(M)$, then fX is $\mathbb{C}$-complete if and only if $f \in I^2(X)$.*

Proof. If X vanishes at some $p \in M$, then so does fX, so the integral curve of fX through p is defined (and constant) for all $t \in \mathbb{C}$. Suppose now that $X(p) \neq 0$. Let $h_p : \mathbb{C} \to R_p(X)$ be the integral curve of X through p. Here, $R_p(X)$ is the orbit of X through p. Then

$$h^*_p(X)(t) = \partial_t,$$

and h_p is a covering map. Since fX is tangent to the orbits of X, $h^*_p(fX)$ is a well-defined vector field on $\mathbb{C}$. Precisely,

$$h^*_p(fX)(t) = f \circ h_p(t)\partial_t.$$

It follows that the integral curve of fX through p is defined for all time if and only if $f \circ h_p(t)$ is an affine linear function of t. This holds for all p in $M\backslash\{X = 0\}$ if and only if $f \in I^2(X)$. □

Proposition 3.2 is a purely holomorphic result. Note that in general, multiplying a (real) vector field by any bounded function preserves completeness.

Example 3.3. Let $\mathbb{C}^n = \mathbb{C} \times \mathbb{C}^{n-1}$ $(n \geq 2)$ have coordinates $z = (z_1, z')$. Consider the vector field ∂_{z_1} on $\mathbb{C}^n$. Then $f(z)\partial_{z_1}$ is complete if and only if $f(z) = g(z') + h(z')z_1$. Vector fields of the form $f(z')\partial_{z_1}$ are called *shear fields*, and those of the form $f(z')z_1\partial_{z_1}$, *overshear fields.* These vector fields have played a fundamental role in the study of automorphisms of $\mathbb{C}^n$, as the set of all time-one maps of these vector fields generates a dense subgroup of Aut($\mathbb{C}^n$) [A, AL].

Definition 3.4. *Let X be a complete holomorphic vector field on a complex manifold M. An X-shear (resp. X-overshear) field on M is a vector field of the form $f \cdot X$, with $f \in I^1(X)$ (resp. $I^2(X)$).*

Second integrals. To find first integrals of a complete vector field X, it is well known that the orbits of X must have particularly nice behavior. Since X maps $I^2(X)$ to $I^1(X)$, we can expect that second integrals are somehow more rare than first integrals. We will show below that this is indeed the case.

One can phrase the problem of finding second integrals, i.e., solving the second order PDE $X^2 f = 0$, as an inhomogeneous first order PDE with conditions on the forcing term:

$$Xv = \varphi \qquad \text{with} \qquad \varphi \in I^1(X). \tag{$*$}$$

The most optimistic situation occurs when we can solve the equation $Xu = 1$. In this case, we can write $f \in I^2(X)$ as

$$f = uXf + (f - uXf),$$

which shows that $I^2(X) = I^1(X) + uI^1(X)$. We shall see, however, that $Xu = 1$ does not always have a solution.

To get a good notion of when $I^2(X)$ is "large", it is convenient to use the language of ideals. Let $J_X := X(I^2(X)) \subset I^1(X)$. J_X is an ideal in $I^1(X)$, since for $\varphi \in I^1(X)$ and $f \in I^2(X)$, $\varphi Xf = X(\varphi f) \in J_X$. Being able to solve $Xu = 1$ is equivalent to saying that $J_X = I^1(X)$. Hence $I^2(X)$ is "large" when $I^1(X)$ is large and the quotient ring $I^1(X)/J_X$ is "small", e.g., finitely generated or trivial.

It is interesting that the size of the quotient $I^1(X)/J_X$ is intimately tied up with the complex geometry of the orbit space of X. Our first result is the following.

Theorem 3.5. *Let $X \in \mathcal{X}_\mathcal{O}(M)$ be complete, $f \in I^2(X)$, and set $N := M \backslash \{Xf = 0\}$. Then*

1. *N/X is a complex manifold,*
2. *$\pi : N \to N/X$ is a holomorphic submersion,*
3. *$\pi \times f : N \to (N/X) \times \mathbb{C}$ is a biholomorphic map, and*
4. $$(\pi \times f)_*(fX)(R_p(X), \lambda) = \psi(R_p(X))\lambda\partial_\lambda$$

   for some $\psi \in \mathcal{O}(N/X)$.

Proof. Let $u := (1/Xf)f$. Then $Xu = 1$, and $u \circ g^t(p) = u(p) + t$. Note also that $X|_N$ is complete, since $\{Xf = 0\}$ is a union of orbits.

1. N/X can be identified with the level set $u^{-1}(0)$ via the map

   $$\xi : N/X \to u^{-1}(0); R_p(X) \mapsto R_p(X) \cap u^{-1}(0).$$

   First, if $p \in N$, then $g_X^{-u(p)}(p) \in u^{-1}(0)$, so that no orbit has empty intersection with $u^{-1}(0)$. Hence ξ is well-defined, at least as a set valued function. Next, note that ξ is single valued. Indeed, if $R_p(X) \cap u^{-1}(0)$ contains p_1 and p_2, then $p_1 = g_X^{t_1}(p)$ and $p_2 = g_X^{t_2}(p)$. But since $u(p_1) = u(p_2)$ and $u \circ g^t(p) = u(p) + t$, we see that $t_1 = t_2$, hence that $p_1 = p_2$. Next, ξ is 1-1 since orbits of vector fields never intersect. Finally, ξ is clearly surjective. To finish 1, note that since $du(X) = 1$, du never vanishes on N. Hence $u^{-1}(0)$ is a complex manifold, which we henceforth identify with N/X via ξ.
2. Observe that the canonical projection $\pi : N \to u^{-1}(0)$ is given by $\pi(p) = g_X^{-u(p)}(p)$. Note also that $\pi|_{u^{-1}(t)} : u^{-1}(t) \to u^{-1}(0)$ is a biholomorphic map; $\pi|_{u^{-1}(t)} = g_X^{-t}$. Hence π is a submersion.
3. Define $\tau : N \times \mathbb{C} \to \mathbb{C}$ and $G : N \times \mathbb{C} \to N$ by

   $$\tau(p, \lambda) := \frac{\lambda - f(p)}{Xf(p)} \qquad \text{and} \qquad G(p, \lambda) := g_X^{\tau(p,\lambda)}(p).$$

   Then, since $f \in I^2(X)$ (and hence $Xf \in I^1(X)$), proposition 3.1 gives that $\tau(g_X^t(p), \lambda) = \tau(p, \lambda) - t$, and hence that

   $$G(g_X^t(p), \lambda) = g_X^{\tau(p,\lambda)-t} \circ g_X^t(p) = G(p, \lambda).$$

   Thus G defines a holomorphic map $H : (N/X) \times \mathbb{C} \to N$ by

   $$H(R_p(X), \lambda) := G(p, \lambda).$$

   Now

   $$\begin{aligned} \pi \times f \circ H(R_p(X), \lambda) &= \pi \times f(g_X^{\tau(p,\lambda)}(p) \\ &= (R_p(X), f(p) + \tau(p, \lambda)Xf(p)) \\ &= (R_p(X), \lambda) \end{aligned}$$

   and

   $$\begin{aligned} H \circ \pi \times f(p) &= H((R_p(X), f(p)) \\ &= g_X^{\tau(p,f(p))}(p) \\ &= p. \end{aligned}$$

   Thus $H = (\pi \times f)^{-1}$, and hence $\pi \times f$ is a biholomorphic map.

4.

$$\begin{aligned} (\pi\times f)_*(X) &= \left.\frac{d}{dt}\right|_{t=0} \pi\times f\circ g_X^t \circ H(R_p(X),\lambda) \\ &= \left.\frac{d}{dt}\right|_{t=0} \pi\times f\circ g_X^t\circ g_X^{\tau(p,\lambda)}(p) \\ &= \left.\frac{d}{dt}\right|_{t=0} (R_p(X), f(p)+\tau(p,\lambda)Xf(p)+tXf(p)) \\ &= Xf(p)\partial_\lambda \end{aligned}$$

So now

$$\begin{aligned} \left((\pi\times f)_*(fX)\right)(R_p(X),\lambda) &= (H^*f)(R_p(X),\lambda)\ \cdot\ (\pi\times f)_*(X)(R_p(X),\lambda) \\ &= \lambda Xf(x)\partial_\lambda \end{aligned}$$

Taking $\psi(R_p(X)) = Xf(p)$ finishes the proof. □

As a corollary, we obtain the following proposition.

Proposition 3.6. *Let $X \in \mathcal{X}_{\mathcal{O}}(M)$ be complete, and define*

$$\Sigma_{X,M} := \bigcap_{f\in I^2(X)} \{Xf=0\}, \qquad N_{X,M} := M\backslash\Sigma_{X,M}.$$

(Note that $N_{X,M}$ is an open subset of M, which is either empty or dense.) Then for each $p \in N_{X,M}$, $R_p(X)$ is biholomorphic to $\mathbb{C}$. In particular, if X has a nontrivial second integral, then almost every orbit of X is biholomorphic to $\mathbb{C}$.

Suppose we can solve $Xv = \varphi \in I^1(X)$. Then theorem 3.5 tells us that N/X is a complex manifold, and N ($= M\backslash\{\varphi = 0\}$) is biholomorphic to $N/X \times \mathbb{C}$. It follows that if M is Stein then N/X is itself Stein (since N is Stein). In the case where $\varphi \equiv 1$, the converse is also true.

Theorem 3.7. *Let $X \in \mathcal{X}_{\mathcal{O}}(M)$ be a complete vector field all of whose orbits are biholomorphic to $\mathbb{C}$. Suppose M/X is a complex manifold and $\pi : M \to M/X$ is a holomorphic map. If M/X is Stein, then $Xu = 1$ has a solution.*

Proof. If M/X is a (differentiable) manifold and π is smooth, then π is a submersion and thus the bundle $\pi : M \to M/X$ is locally trivial. Furthermore, it is possible to select local trivializations $\{\varphi_j : \pi^{-1}(U_j) \to U_j\times\mathbb{C}\}$ such that $(\varphi_j)_*X = \partial_\lambda$ for all j. Indeed, let σ_j be a local section of $\pi : M \to M/X$ over U_j. For each $x \in \pi^{-1}(U_j)$, define $\lambda = \lambda(x)$ to be the unique complex number for which $g_X^\lambda(\sigma_j\circ\pi(x)) = x$. λ is holomorphic because of the holomorphic dependence of the flow on initial conditions. Set $\varphi_j(x) := (\pi(x), \lambda(x))$. Note that $\varphi_j \circ g_X^s(x) = (\pi(x), s+\lambda(x))$ and so

$$(\varphi_j)_*X(x) = \left.\frac{d}{ds}\right|_{s=0} (\pi(x), s+\lambda(x)) = \partial_\lambda.$$

Now, since the fibers of our holomorphic bundle are $\mathbb{C}$, the bundle must be an affine bundle, and so the transition functions $\varphi_{jk}(\pi(x))t := \mathrm{pr}_\lambda \circ \varphi_j \circ \varphi_k^{-1}(x,t)$ (where pr_λ is the projection to the second factor) satisfy

$$\varphi_{jk}(\pi(x))t = f_{jk}(\pi(x))t + g_{jk}(\pi(x)).$$

Moreover, because of the way the φ_j were chosen, $f_{jk}(\pi(x)) \equiv 1$ for all j,k. Indeed,

$$f_{jk}(\pi(x)) = \frac{\partial}{\partial t}\varphi_{jk}(\pi(x))t = \mathrm{pr}_{\lambda *}(\varphi_j)_*(\varphi_k^{-1})_*\partial_\lambda \equiv 1.$$

Next, writing out the identity

$$\varphi_{jk}\circ\varphi_{kl}\circ\varphi_{lj} = id$$

shows that $\{g_{jk}\}$ is a 1-cocycle on M/X, i.e., Cousin-1 data. Since M/X is Stein, $g_{jk} = g_j - g_k$. One checks easily that $\{g_k\}$ is a section of $\pi : M \to M/X$. It follows that $\pi : M \to M/X$ is actually a line bundle, since we can use the section $\{g_k\}$ as an origin for each fiber. Precisely, we can define the transition functions

$$P_{jk}(\pi(x))v := \varphi_{jk}(\pi(x))(v + g_k(\pi(x))) - g_j(\pi(x)).$$

Then $P_{jk}(\pi(x))v = f_{jk}(\pi(x))(v) + \varphi_{jk}(\pi(x))(g_k(\pi(x))) - g_j(\pi(x)) = f_{jk}(\pi(x))(v)$ so that, since $f_{jk} \equiv 1$, $\pi : M \to M/X$ is trivial. We now define (in the usual way) the global trivialization $F : M \to (M/X) \times \mathbb{C}$ by $F := \pi \times \psi$, where

$$\psi(x) = \mathrm{pr}_\lambda \circ \varphi_j(x) - g_j(\pi(x)) \qquad \text{for } x \in \pi^{-1}(U_j).$$

ψ is well defined, since for $x \in U_j \cap U_k$,

$$\begin{aligned} \mathrm{pr}_\lambda \circ \varphi_j(x) - g_j(\pi(x)) &= \mathrm{pr}_\lambda \circ \varphi_j \circ \varphi_k^{-1}(\varphi_k(x)) - g_j(\pi(x)) \\ &= \varphi_{jk}(\pi(\varphi_k(x))t - g_j(\pi(x)) \qquad \text{where } t = \mathrm{pr}_\lambda(\varphi_k(x)) \\ &= t + g_{jk}(\pi(x)) - g_j(\pi(x)) \\ &= t - g_k(\pi(x)) \\ &= \mathrm{pr}_\lambda \circ \varphi_k(x) - g_k(\pi(x)) \end{aligned}$$

It follows that

$$F_* X = \partial_\lambda.$$

Setting $u(F^{-1}(\pi(x), \lambda)) = \lambda$, we see that $Xu = X(F^*(F_* u)) = (F_* X)(F_* u) = 1$, as required. ☐

Remark: A more careful look at the proof shows that one does not need M/X to be Stein, but only that $H^1(M/X, \mathcal{O}) = 0$.

Remark: Theorems 3.5 and 3.7 explain in part why it was so much easier to prove density theorems for spaces of the form $M \times \mathbb{C}$.

Example 3.8. Let

$$X(x) = a\partial_b + c\partial_d \in \mathcal{X}_{\mathcal{O}}(Sl(2,\mathbb{C})) \qquad x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sl(2,\mathbb{C}).$$

X is a left invariant vector field on $Sl(2,\mathbb{C})$ whose orbits are closed and biholomorphic to $\mathbb{C}$. Hence $Sl(2,\mathbb{C})/X$ is a complex manifold. Nevertheless the equation $Xu = 1$ has no global holomorphic solutions[1]. Indeed, $Sl(2,\mathbb{C})$ is homotopy equivalent (by the Gram-Schmidt algorithm) to $SU(2) \cong S^3$, which is a cell complex of dimension 3. It follows that $Sl(2,\mathbb{C})$ is not biholomorphic to $B \times \mathbb{C}$, for then B would be a Stein 2-fold with 3-dimensional cells, a contradiction.

4. (M^2, ω)

To recall, we define

$$M^2 = \mathbb{C}^2 \setminus \{xy = 1\} \qquad \text{and} \qquad \omega = \frac{1}{xy - 1} dx \wedge dy.$$

In this section we prove Theorem 1.

Notation and facts. It will be convenient to write $z = xy - 1$. As we mentioned before, M^2 admits two everywhere independent complete vector fields

$$X(x,y) := z\partial_y \qquad \text{and} \qquad Y(x,y) := z\partial_x.$$

Since z does not vanish on M^2, it is clear that every holomorphic vector field on M^2 is of the form $fX + gY$ for some $f, g \in \mathcal{O}(M^2)$. We note also that

$$H(x,y) = x\partial_x - y\partial_y$$

is a complete holomorphic vector field with zero divergence. One can integrate X, Y and H to see that every orbit of X and Y is biholomorphic to $\mathbb{C}^*$ and that this is also the case for every orbit of H except for its

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[1] It is interesting to note, however, that $u = (\bar{a}b + \bar{c}d)/(|a|^2 + |c|^2)$ is a real analytic solution, and that the $\mathbb{C}$-fibration $Sl(2,\mathbb{C}) \to Sl(2,\mathbb{C})/X$ is real-analytically trivial

single fixed point at the origin of $\mathbb{C}^2$. Hence X, Y and H have no nontrivial second integrals. The following facts are easily computed:

$$\begin{array}{ccccc} [H,X]=X & [H,Y] & = & -Y & [X,Y]=zH \\ & xY-yX & = & zH & \\ Xx=0 & Xy & = & z & Xz=xz \\ Yx=z & Yy & = & 0 & Yz=yz \\ Hx=x & Hy & = & -y & Hz=0 \end{array}$$

Lemma 4.1. *Every $\varphi \in \mathcal{O}(M^2)$ is of the form*

$$\varphi(x,y)=f(x,y,z)$$

for some $f \in \mathcal{O}(\mathbb{C}^2 \times \mathbb{C}^*)$.

Proof. $j:(x,y)\mapsto(x,y,z)$ gives a proper holomorphic embedding of M^2 into $\mathbb{C}^2\times\mathbb{C}^*$. It is thus a standard fact (Theorem A) that $\mathcal{O}(M^2)=\mathcal{O}(\mathbb{C}^2\times\mathbb{C}^*)|_{M^2}$. □

Thus the Laurent polynomials

$$\sum_{j,l\geq 0} c_{jl}x^jz^{-l}+\sum_{k,l\geq 0} d_{kl}y^kz^{-l}+\sum_{l\geq 0} e_lz^{-l}+\sum_{k,l\geq 0} f_{jk}x^jy^k$$

are dense in $\mathcal{O}(M^2)$. We shall call such Laurent polynomials *reduced.*

The key lemmas.

Lemma 4.2. *Let j and k be nonnegative integers. Then for some polynomial $p(x,y)$ there is a divergence zero completely generated vector field of the form*

$$x^jy^kX+p(x,y)Y$$

Proof. Since $Xx=0$, x^jX is complete, which proves the claim for $k=0$. Note next that, since $Yy=0$, y^lY is complete, and hence (as a computation shows)

$$[y^kY,x^jX]=(j+1)x^jy^{k+1}X-jx^{j-1}y^kX+p_1(x,y)Y$$

is divergence zero completely generated. The result follows by induction on k. □

This lemma has a corollary which is of independent interest. Let $\mathfrak{g}$ denote the Lie algebra of all holomorphic vector fields of $\mathbb{C}^2$ which vanish on $\{xy=1\}$, and have ω-divergence zero.

Corollary 4.3. *$\mathfrak{g}$ has the density property.*

Proof. Note first that the set of divergence zero vector fields of the form $p(x,y)X+q(x,y)Y$ for polynomials p and q, is dense in $\mathfrak{g}$. Let V be one such vector field. By lemma 4.2 there exists another such vector field W which is completely generated, such that $V-W=p_1(x,y)Y$. But since $0=div(V-W)=Y(p_1)$, $V-W$ is complete. Thus $V=W+(V-W)$ is completely generated, as desired. □

The following identities are just computations, the last two most easily proved using the commutation relations given above. We omit the details.

$$\begin{aligned} -z^{-l}H &= yz^{-l}X+(*)Y \\ \left[z^{-l}H,y^kY\right] &= ly^{k+2}z^{-(l+1)}X+(*)Y \\ \left[x^jX,z^{-l}H\right] &= lx^jz^{-(l+1)}X+(l-j-1)x^jz^{-l}X+(*)Y \end{aligned}$$

Here and below, the symbol $(*)$ means a polynomial in x, y and $1/z$. Using the first identity we have

Lemma 4.4. *For each $l\geq 1$ there exists a complete divergence zero vector field of the form*

$$yz^{-l}X+(*)Y.$$

Using the second identity we have

Lemma 4.5. *For each $l \geq 2$ and $k \geq 0$ there exists a divergence zero completely generated vector field of the form*

$$y^{k+2}z^{-l}X + (*)Y.$$

Using the third identity we have by induction

Lemma 4.6. *For each $l \geq 2$ and $j \geq 0$ there exists a polynomial $p(x)$ and a divergence zero completely generated vector field of the form*

$$y^j z^{-l}X + p(x)z^{-1}X + (*)Y.$$

Lemma 4.7. *Suppose that p and q are polynomials in one variable, that $g \in \mathcal{O}(M^2)$, and that*

$$V(x,y) = \frac{p(x) + yq(y)}{z}X + g(x,y)Y$$

is a divergence zero vector field. Then $p = 0$ and q is constant.

Proof. The vanishing divergence of V is equivalent to the closedness of the holomorphic one form $\theta = i_V\omega$. An easy computation shows that

$$\theta = -\frac{p(x) + yq(y)}{z}dx + g(x,y)dy.$$

It follows from Stokes' theorem that if Ω is a smooth 2-manifold with boundary, then

$$\int_{\partial\Omega} \theta = 0.$$

For $y \in \mathbb{C}^*$, let $\gamma_y : [0, 2\pi] \to M^2$ be defined by

$$\gamma_y(t) = \left((e^{it} + 1)y, y^{-1}\right).$$

Note that

$$\begin{aligned} \int_{\gamma_y} \theta &= \int_0^{2\pi} \frac{p((1 + e^{it})y) + (1/y)q(1/y)}{e^{it}} iye^{it}dt \\ &= 2\pi i(yp(y) + q(1/y)) \end{aligned}$$

Fix y_0 and y_1 in $\mathbb{C}^*$, and let $\beta : [0,1] \to \mathbb{C}^*$ be any smooth curve with $\beta(0) = y_0$ and $\beta(1) = y_1$. Then

$$\Omega_{y_0,y_1} := \{\gamma_{\beta(s)}(t) \mid (t,s) \in [0,1] \times [0,2\pi]\}$$

is a smooth cylinder in M^2, and

$$\partial\Omega = \gamma_{y_0} \cup \gamma_{y_1}.$$

Since y_0 and y_1 were arbitrary, it follows that the Laurent polynomial $yp(y) + q(1/y)$ is constant, and hence that $p = 0$ and q is constant. This completes the proof. □

Proof of theorem 2. Let $V = fX + gY$ be a holomorphic vector field with f and g reduced (see the remark following lemma 4.1) Laurent polynomials. By lemmas 4.2 , 4.5 and 4.6 there exists a divergence zero completely generated vector field W_1 so that $V - W_1 = ((p(x) + yq(y))/z)X + (*)Y$. According to lemma 4.7 , $p = 0$ and q is constant. Hence by lemma 4.4 there is a complete vector field W_2 such that $V - W_1 - W_2 = h(x,y)Y$ for some $h \in \mathcal{O}(M^2)$. But since $Yh = div(hY) = 0$, $V - W_1 - W_2$ is complete. Hence

$$V = W_1 + W_2 + (V - W_1 - W_2)$$

is divergence zero completely generated, as desired. □

As mentioned in the introduction, it is not known whether there exists a single automorphism f of M^2 such that $f^*\omega \neq \pm\omega$. However, this difficulty is immediately lifted by "stabilizing" M^2. Then theorem 2 and main result I.3 in [V1] imply the following

Corollary 4.8. *$M^2 \times \mathbb{C}$ has the density property.*

5. SL(2,C)

In this section we will prove Theorem 2.

Notation and facts. $Sl(2,\mathbb{C})$ will be represented as the set of all 2×2 matrices with complex entries having determinant 1. We will write the members of $Sl(2,\mathbb{C})$ as

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \text{with } ad - bc = 1.$$

We shall use a, b, c, d as coordinates on $\mathbb{C}^4$, in which we will think of $Sl(2,\mathbb{C})$ as a submanifold. The canonical basis of left invariant vector fields will be employed throughout. These are

$$X(a,b,c,d) = a\partial_b + c\partial_d, \quad Y(a,b,c,d) = b\partial_a + d\partial_c, \quad \text{and}$$
$$H(a,b,c,d) = a\partial_a - b\partial_b + c\partial_c - d\partial_d.$$

The relevant commutation relations are

$$[H,X] = 2X, \quad [H,Y] = -2Y, \quad \text{and} \quad [X,Y] = H.$$

Of course, X Y and H are $\mathbb{C}$-complete, being left invariant.

Since X, Y and H trivialize the tangent bundle of $Sl(2,\mathbb{C})$, an arbitrary vector field $V \in \mathcal{X}_{\mathcal{O}}(Sl(2,\mathbb{C}))$ may be written

$$V = V_X X + V_Y Y + V_H H, \qquad V_X, V_Y, V_H \in \mathcal{O}(Sl(2,\mathbb{C})).$$

We then define

$$div(V) := XV_X + YV_Y + HV_H.$$

The operator $div : \mathcal{X}_{\mathcal{O}}(Sl(2,\mathbb{C})) \to \mathcal{O}(Sl(2,\mathbb{C}))$ is, up to a constant, the usual divergence operator associated to any left invariant holomorphic 3-form on $Sl(2,\mathbb{C})$. Consequently, it satisfies, for any holomorphic function f and vector fields U and V,

(i) linearity,
(ii) $div(fV) = Vf + f divV$, and
(iii) $div[U,V] = U divV - V divU$.

We shall also have occasion to use the right invariant vector fields on $Sl(2,\mathbb{C})$. The canonical basis is

$$x = c\partial_a + d\partial_b, \quad y = a\partial_c + b\partial_d, \quad \text{and} \quad h = a\partial_a + b\partial_b - c\partial_c - d\partial_d.$$

It is useful to note that

$$\begin{aligned} x &= d^2X - c^2Y + cdH \\ y &= -b^2X + a^2Y - abH, \qquad \text{and} \\ h &= 2bdX - 2acY + (ad+bc)H \end{aligned}$$

Finally,

$$I^1(X) = \langle a, c\rangle, \quad I^1(Y) = \langle b, d\rangle, \quad \text{and} \quad I^1(H) = \langle a^m b^k c^n d^l | m + n - k - l = 0\rangle.$$

Since every orbit of H is biholomorphic to $\mathbb{C}^*$, $I^2(H) = I^1(H)$. For X and Y, the relevant facts about I^2 are that $Xb = a$ and $Xd = c$, and that $Ya = b$ and $Yc = d$. We will not need anything about the second integrals of right invariant vector fields, but we will use the facts that $I^1(x) = \langle c, d\rangle$, $I^1(y) = \langle a, b\rangle$, and $I^1(h) = \langle a^m b^k c^n d^l | m + k - n - l = 0\rangle$.

The volume density property. The volume density property for $Sl(2,\mathbb{C})$ follows immediately from the following.

Theorem 5.1. *Every divergence zero polynomial vector field on $Sl(2,\mathbb{C})$ is divergence zero completely generated.*

We shall now prove this theorem. The proof involves many steps, and must be broken up into cases. These cases are isolated according to certain values of an index of monomials. We call this index the H-index, and define it as

$$ind_H(a^m b^k c^n d^l) := m - k + n - l.$$

Note that $H(a^m b^k c^n d^l) = ind_H(a^m b^k c^n d^l) a^m b^k c^n d^l$. A polynomial in a, b, c and d will be called H-homogeneous of degree r if the H-index of each of its monomials is r. We note that H-homogeneous polynomials is a concept which descends to $Sl(2,\mathbb{C})$, i.e., when we identify $ad - bc$ and 1. Let us further point out that while nonzero constants have H-index 0, 0 has every integer as its H-index. Finally, note that X raises the H-index of an H-homogeneous polynomial by 2, and that Y lowers the H-index of an H-homogeneous polynomial by 2:

$$\begin{aligned} H(X(a^m b^k c^n d^l)) &= XH(a^m b^k c^n d^l) + [H,X](a^m b^k c^n d^l) \\ &= (ind_H(a^m b^k c^n d^l) + 2)X(a^m b^k c^n d^l) \end{aligned}$$

and

$$\begin{aligned} H(Y(a^m b^k c^n d^l)) &= YH(a^m b^k c^n d^l) + [H,Y](a^m b^k c^n d^l) \\ &= (ind_H(a^m b^k c^n d^l) - 2)Y(a^m b^k c^n d^l). \end{aligned}$$

Finally, we leave it to the reader to check that completeness holds where necessary.

Lemma 5.2. *Let $a^m b^k c^n d^l$ be a monomial of H-index different from -2. Then there exists a completely generated polynomial vector field of the form*

$$a^m b^k c^n d^l X + p(a,b,c,d)H.$$

We shall simultaneously prove

Lemma 5.3. *Let $a^m b^k c^n d^l$ be a monomial of H-index different from 2. Then there exists a completely generated polynomial vector field of the form*

$$a^m b^k c^n d^l Y + p(a,b,c,d)H.$$

Proof. We mark the end of proof of each case by the symbol Δ.
Case 1 $(X, ind_H \geq 0)$: Let m_1, m_2, n_1 and n_2 be nonnegative integers such that $m_1 - k + n_1 - l = 0$, $m_1 + m_2 = m$, and $n_1 + n_2 = n$. Then

$$[a^{m_1} b^k c^{n_1} d^l H, a^{m_2} c^{n_2} X] = (m_2 + n_2 + 2) a^m b^k c^n d^l X + pH.$$

Δ

Case 2 $(Y, ind_H \leq 0)$: Let k_1, k_2, l_1 and l_2 be nonnegative integers such that $m - k_1 + n - l_1 = 0$, $k_1 + k_2 = k$, and $l_1 + l_2 = l$. Then

$$[b^{k_2} d^{l_2} Y, a^m b^{k_1} c^n d^{l_1} H] = (k_2 + l_2 + 2) a^m b^k c^n d^l Y + pH.$$

Δ

In the remaining cases, the following identities will be very useful:

$$\begin{aligned} [a^m b^{k_1} c^n d^{l_1} H, [b^{k_2} d^{l_2} Y, aX]] &= [a^m b^{k_1} c^n d^{l_1} H, b^{k_2+1} d^{l_2} X \\ &\qquad - a(k_2 ad + l_2 bc) b^{k_2-1} d^{l_2-1} Y + pH] \\ &= (1 - k_2 - l_2)\left(a^m b^{k_1+k_2+1} c^n d^{l_1+l_2} X - \right. \\ &\qquad \left. a(k_2 ad + l_2 bc) a^m b^{k_1+k_2-1} c^n d^{l_1+l_2-1} Y\right) \\ &\qquad\qquad + pH \end{aligned}$$

$$\begin{aligned} [a^m c^n d^{l_1} H, [d^{l_2} Y, cX]] &= [a^m c^n d^{l_1} H, d^{l_2+1} X - l_2 c^2 d^{l_2-1} Y + pH] \\ &= (1 - l_2)\left(a^m c^n d^{l_1+l_2+1} X - l_2 a^m c^{n+2} d^{l_1+l_2-1} Y\right) \\ &\qquad + pH \end{aligned}$$

We will refer to these as identity 1 and identity 2 respectively.

case 3 $(X, ind_H \leq -4, k > 0)$: Let $k_1, k_2, l_1, l_2 \geq 0$ be such that $m - k_1 + n - l_1 = 0$, $k = k_1 + k_2 + 1$ and $l = l_1 + l_2$. Since $m - k + n - l \leq -4$, $1 + m - (k_1 + k_2 - 1) + n - (l_1 + l_2 - 1) \leq 0$. Thus, using identity 1, we can, via case 2, eliminate the Y component. Δ

case 4 $(X, ind_H \leq -4, k = 0)$: Let $l_1, l_2 \geq 0$ be such that $m + n - l_1 = 0$ and $l = l_1 + l_2 + 1$. Since $m + n - l \leq -4$, $m + (n+2) - (l_1 + l_2 - 1) \leq 0$. Thus, using identity 2, we can, again via case 2, eliminate the Y component. Δ

case 5 ($Y, ind_H \geq 4$): This case can be handled as cases 3 and 4, using appropriate modifications of the identities 1 and 2, and using case 1 instead of case 2. Specifically, one interchanges the roles of X and Y, of a and d, of b and c, of m and l, and of n and k. The details are left to the interested reader. Δ

case 6 ($X, ind_H = -1, k > 0$): With $k_2 = l_2 = 0$, identity 1 takes the form

$$[a^m b^{k_1} c^n d^l H, [Y, aX]] = a^m b^{k_1+1} c^n d^l X + pH.$$

Letting $k = k_1 + 1$ finishes this case. Δ

case 7 ($X, ind_H = -1, k = 0$): With $l_2 = 0$, identity 2 takes the form

$$[a^m b^{k_1} c^n d^l H, [Y, aX]] = a^m b^k c^n d^{l_1+1} X + pH.$$

Letting $l = l_1 + 1$ finishes this case. Δ

case 8 ($Y, ind_H = 1$): Again, just use calculations analogous to those of cases 6 and 7. Δ

case 9 ($X, ind_H = -3$): Using identities 1 and 2, and case 8, we can eliminate the Y components, which have H-index 1. Notice that in this case, $1 - k_2 - l_2 \neq 0$. Δ

case 10 ($Y, ind_H = 3$): This case is analogous to case 9. Δ

This completes the proof. $\square$

Lemmas 5.2 and 5.3 become false if the index conditions are removed. Fortunately this is not necessary to proceed.

Lemma 5.4. *Let $a^m b^k c^n d^l$ be an index -2 monomial. Then there exists a completely generated divergence zero polynomial vector field V of the form*

$$V = a^m b^k c^n d^l X + (*)Y + (*)H.$$

Proof. First, let us call a monomial $a^m b^k c^n d^l$ (a,d)-reduced if either m or l are zero. Every polynomial p on $Sl(2,\mathbb{C})$ can be written uniquely as a linear combination of (a,d)-reduced monomials. Furthermore, p is (a,d)-reduced if and only if for every left invariant vector field L, Lp is (a,d)-reduced.

case 1 ($l > 0, m = 0$): Here $V = b^k c^n d^l X + (*)Y + (*)H$. We thus need only note that

$$\frac{1}{n+1}[b^k d^{l-1} Y, c^{n+1} X] = b^k c^n d^l X + (*)Y + (*)H.$$

This finishes case 1. Δ

case 2 ($l = 0$): We may assume that $V = a^m b^k c^n X - pY + (*)H$, where p is an (a,d)-reduced, H-homogeneous polynomial of H-index 2. Now,

$$0 = divV = ka^{m+1} b^{k-1} c^n - Yp,$$

and so $Yp = ka^{m+1} b^{k-1} c^n$. Note thus, that every (a,d)-reduced monomial component of p must be of the form $a^{m'} b^{k'}$. It follows that $n = 0$, and that p is a monomial, which must be $Ca^{m'} b^{k'}$. The index conditions then become $m' = k' + 2$ and $m = k - 2$. Next, comparing exponents of Yp and $a^{m+1} b^{k-1}$, we see that $m' = m + 2$. Hence if $divV = 0$, then V is restricted to be of the form

$$V = (ab)^m b^2 X - (ab)^m a^2 Y + (*)H.$$

It follows that

$$V + (ab)^m y = (*)H,$$

and hence V is in fact complete mod H. This finishes case 2, and the proof of the lemma. $\square$

Lemma 5.5. *Let p be a non-zero, H-homogeneous polynomial of H-index 2. Then there is no divergence zero vector field of the form $pY + qH$.*

Proof. Since Yp is of H-index 0, so is Hq. But as H preserves H-index, q is of H-index 0. Hence $Hq = 0$, so that $Yp = 0$. But every nonzero first integral of Y has nonpositive H-index. Since p is of H-index 2, it must vanish identically. □

Proof of theorem 5.1: Let V be a polynomial vector field of zero divergence. By lemmas 5.2, 5.3 and 5.4, there is a divergence zero completely generated vector field W, such that $V - W = pY + qH$, where p is an H-homogeneous polynomial of H-index 2. By lemma 5.5, $p = 0$. Thus $Hq = 0$, and so qH is complete. We see that $V = W + qH$ is divergence zero completely generated, as desired. □

The divergence lemma.

Lemma 5.6. *Let $V \in \mathcal{X}_{\mathcal{O}}(Sl(2,\mathbb{C}))$ be a polynomial vector field. Then there exists a completely generated polynomial vector field $W \in \mathcal{X}_{\mathcal{O}}(Sl(2,\mathbb{C}))$ such that*

$$divW = divV.$$

Proof. The image by div of the polynomial vector fields is spanned by the following polynomials:

(i) $div(a^m b^k c^n d^l X) = a^m c^n X(b^k d^l)$,
(ii) $div(a^m b^k c^n d^l Y) = b^k c^l Y(a^m c^n)$, and
(iii) $div(a^m b^k c^n d^l H) = H(a^m b^k c^n d^l)$.

Here m, k, n and l range over all nonnegative integers. We need only to show that each of these polynomials is the image by div of a completely generated vector field. To this end, observe that

$$div\left[a^m c^n X, \frac{1}{k+l}(kab^{k-1}d^l + lb^k cd^{l-1})Y\right] = a^m c^n X(b^k d^l),$$

and that

$$div\left[b^k d^l Y, \frac{1}{m+n}(ma^{m-1}bc^n + na^m c^{n-1}d)X\right] = b^k c^l Y(a^m c^n).$$

This takes care of cases (i) and (ii). Case (iii) is only slightly more detailed. To handle it, let $j = m+n-k-l$. If $j = 0$ then $H(a^m b^k c^n d^l) = 0$ so there's nothing to do. Suppose that $j > 0$. Let m_1, m_2, n_1 and n_2 be nonnegative integers such that

(a) $m = m_1 + m_2$ and $n = n_1 + n_2$, and
(b) $m_1 - k + n_1 - l = 0$.

It follows that $m_2 + n_2 = j$. Then

$$\begin{aligned} div & \left[a^{m_1} b^k c^{n_1} d^l H, \frac{1}{m_2+n_2}(m_2 a^{m_2-1} bc^{n_2} + n_2 a^{m_2} c^{n_2-1} d)X\right] \\ &= a^{m_1} b^k c^{n_1} d^l H(a^{m_2} c^{n_2}) \\ &= H(a^m b^k c^n d^l). \end{aligned}$$

Finally, if $j < 0$, let k_1, k_2, l_1 and l_2 be nonnegative integers such that

(a) $k = k_1 + k_2$ and $l = l_1 + l_2$, and
(b) $m - k_1 + n - l_1 = 0$.

Then $k_2 + l_2 = -j$ and we have

$$\begin{aligned} div & \left[a^m b^{k_1} c^n d^{l_1} H, \frac{1}{k_2+l_2}(k_2 ab^{k_2-1} d^{l_2} + l_2 b^{k_2} cd^{l_2-1})Y\right] \\ &= a^m b^{k_1} c^n d^{l_1} H(b^{k_2} d^{l_2}) \\ &= H(a^m b^k c^n d^l). \end{aligned}$$

The reader may confirm directly or via the ideas in section 3 that all of the vector fields used were complete where required. This completes the proof. □

Proof of theorem 2. Let $U \in \mathcal{X}_{\mathcal{O}}(Sl(2,\mathbb{C}))$ be a polynomial vector field. By lemma 5.6 there exists a completely generated vector field $U' \in \mathcal{X}_{\mathcal{O}}(Sl(2,\mathbb{C}))$ which is polynomial, such that $divU = divU'$. Since $V := U - U'$ is a polynomial vector field with zero divergence, it is, by theorem 5.1, completely generated. Hence $U = U' + V$ is completely generated, and Theorem 2 now follows from the density of polynomial vector fields in $\mathcal{X}_{\mathcal{O}}(Sl(2,\mathbb{C}))$. □

6. ELLIPTIC MICROSPRAY MANIFOLDS

In this section we explore more fully the density and volume density property on spaces of the form $M \times \mathbb{C}$. The case in which M is a complex Lie group was already handled in our note [V1]. The proofs of the density theorems in this section are very similar to those in the less general case [V1], and thus will be very sketchy. The main point here is to broaden the class of such complex manifolds M in hopes of giving insight into the density and volume density property.

Definitions and examples.

Definition 6.1. *An Elliptic Microspray (EM) manifold is a complex manifold M with the property that for any $V \in \mathcal{X}_{\mathcal{O}}(M)$, compact $K \Subset M$ and $\epsilon > 0$ there are functions $f_1, ..., f_r \in \mathcal{O}(K)$ and $\mathbb{C}$-completely generated vector fields $X_1, ..., X_r$ satisfying*

$$\left|\left|V - \sum f_j X_j\right|\right|_K < \epsilon$$

It is also useful to consider slightly more restrictive structures.

Definition 6.2. *An EMV (V for volume) manifold is a pair (M, ω), where M is a complex manifold and ω is a holomorphic volume element on M, with the property that for any $V \in \mathcal{X}_{\mathcal{O}}(M)$, compact $K \Subset M$ and $\epsilon > 0$ there are functions $f_1, ..., f_r \in \mathcal{O}(K)$ and divergence zero completely generated vector fields $X_1, ..., X_r$ satisfying*

$$\left|\left|V - \sum f_j X_j\right|\right|_K < \epsilon$$

Of course, every EMV manifold is EM. The terminology we have chosen is inspired by that in [Gro].

Examples

1. Every complex Lie group G is EMV. Indeed, the left invariant vector fields, which are all complete, parallelize the tangent bundle of G, so every vector field can be written in the form $\sum f_j V_j$ where $f_j \in \mathcal{O}(G)$ and $\{V_j\}$ is any fixed basis of $\mathfrak{g} = \mathcal{L}ie(G)$. Moreover, $div \sum f_j V_j = \sum V_j f_j$, so every left invariant vector field has zero divergence.
2. Every Stein complex homogeneous space is EMV. Indeed, let G be a complex Lie group, and H a closed complex subgroup such that $M = H \backslash G = \{Hg \; ; \; g \in G\}$ is Stein. The left invariant vector fields on G will project to M, as will the left invariant k-forms ($k = dim_{\mathbb{C}} M$). Let V be the vector space spanned by the projection to M of the left invariant vector fields on G. All of these vector fields have divergence zero with respect to any non-zero volume element coming from a left invariant k-form on G. Our claim is then proved if we can show that $\mathcal{X}_{\mathcal{O}}(M) = \mathcal{O}(M) \otimes V$. To see the latter, consider the following short exact sequence of coherent sheaves on M:

   $$0 \to \mathcal{S} \to \mathcal{O} \otimes V \to \mathcal{X}_{\mathcal{O}} \to 0.$$

   Here $\mathcal{O}$ is the structure sheaf and $\mathcal{X}_{\mathcal{O}}$ is the tangent bundle sheaf. The sequence gives rise to a long exact sequence in cohomology, a portion of which is

   $$H^0(\mathcal{O} \otimes V, M) \to H^0(\mathcal{X}_{\mathcal{O}}, M) \to H^1(\mathcal{S}, M).$$

   Since M is Stein, $H^1(\mathcal{S}, M) = 0$ and our claim follows.

Density theorems. Our first result is a stable volume density property theorem for EMV manifolds.

Theorem 6.3. *If (M, ω) is an EMV manifold, then $(M \times \mathbb{C}, \omega \wedge dz)$ has the volume density property.*

Proof. First, let $V = z^n X + (*)\partial_z$, $X \in \mathcal{X}_{\mathcal{O}}(M)$ be a divergence zero vector field. We can assume (by approximation) that $X = \sum \varphi_j Y_j$ with $Y_j \in \mathcal{X}_{\mathcal{O}}(M, \omega)$ divergence zero $\mathbb{C}$-completely generated. Now

$$[1/(n+1)z^{n+1}Y_j, \varphi_j \partial_z] = z^n \varphi_j Y_j + (*)\partial_z$$

is clearly divergence zero $\mathbb{C}$-completely generated, and hence V is divergence zero $\mathbb{C}$-completely generated modulo ∂_z. That is, there exists a holomorphic vector field W which is divergence zero $\mathbb{C}$-completely generated, and has the property that $V - W = \psi(x, z)\partial_z$. But then $0 = div(V - W) = \partial_z \psi$, so that $V - W$ is complete. Hence $V = W + (V - W)$ is divergence zero $\mathbb{C}$-completely generated. Since every divergence zero vector field can be approximated by sums of vector fields of the same form as V, we are done. ☐

The next result is that EM manifolds with holomorphic volume elements are stably EMV.

Proposition 6.4. *If M is an EM manifold and ω is a nonvanishing holomorphic volume element on M, then $(M \times \mathbb{C}, \omega \wedge dz)$ is EMV.*

We shall need the following lemma.

Lemma 6.5. *Let M and ω be as in proposition 6.4. If $X \in \mathcal{X}_{\mathcal{O}}(M)$ is ($\mathbb{C}$-) completely generated, then there exists $\tilde{X} \in \mathcal{X}_{\mathcal{O}}(M \times \mathbb{C}, \omega \wedge dz)$ which is divergence zero completely generated, such that*

$$\tilde{X} - X = (*)\partial_z.$$

Proof. First note that if $X \in \mathcal{X}_{\mathcal{O}}(M)$ is complete, then so is $X - z(div_\omega X)\partial_z$. Moreover, the latter has zero $\omega \wedge dz$ divergence. Next notice that $X + (*)\partial_z + Y + (*)\partial_z = X + Y + (*)\partial_z$ and that $[X + (*)\partial_z, Y + (*)\partial_z] = [X, Y] + (*)\partial_z$. The lemma follows easily from these facts. ☐

Proof of proposition 6.4: Let $X \in \mathcal{X}_{\mathcal{O}}(M \times \mathbb{C}, \omega \wedge dz)$ be written as $X = \sum z^j V_j + (*)\partial_z$ where $V_j \in \mathcal{X}_{\mathcal{O}}(M)$. By approximation, we may assume that the sum is finite. Since M is EM, we may write (again, up to approximation) $V_j = \sum_k f_{jk} S_{jk}$ where $S_{jk} \in \mathcal{X}_{\mathcal{O}}(M)$ are $\mathbb{C}$-completely generated. Now, for each S_{jk} the lemma guarantees a divergence zero completely generated $\tilde{S}_{jk} \in \mathcal{X}_{\mathcal{O}}(M \times \mathbb{C}, \omega \wedge dz)$ so that $S_{jk} - \tilde{S}_{jk} = (*)\partial_z$. It follows that (up to approximation)

$$X = \sum_{jk} z^j f_{jk} \tilde{S}_{jk} + (*)\partial_z,$$

which is exactly what was needed. ☐

Using main result I.3 in [V1], one immediately obtains the following.

Corollary 6.6. *If M is a Stein EMV space, then $M \times \mathbb{C}$ has the density property. If M is a Stein EM space and M admits a holomorphic volume element, then $M \times \mathbb{C}^2$ has the density property.*

7. A QUESTION

The results in section 6 suggest the following natural question:

Is there a difference between the volume density property and EMV?

To date, in all the examples for which we have been able to settle this question, the answer is "no". If the answer "no" can be established in general, this would represent a major breakthrough. However, it is by no means clear what the answer is. Again, one needs candidates for testing. We propose one now. Let

$$\Sigma^3 := \{(a, b, c, d) \in \mathbb{C}^4 \mid a^2 d - bc = 1\}.$$

Σ^3 is a smooth subvariety of $\mathbb{C}^4$ and is also a branched double cover of $SL(2, \mathbb{C})$. Moreover, Σ^3 admits some interesting complete vector fields:

$$X = a^2 \partial_b + c\partial_d, \quad Y = b\partial_a + 2ad\partial_c \quad \text{and} \quad H = a\partial_a - 2b\partial_b + 2c\partial_c - 2d\partial_d$$

correspond to the left invariant vector fields of $SL(2,\mathbb{C})$, and

$$\xi = a^2\partial_c + b\partial_d, \quad \eta = c\partial_a + 2ad\partial_b \quad \text{and} \quad \theta = a\partial_a + 2b\partial_b - 2c\partial_c - 2d\partial_d$$

correspond to the right invariant vector fields of $SL(2,\mathbb{C})$. Since Σ^3 is three dimensional, we expect some relations between the left and right vector fields. A calculation shows that

$$\xi = -b^2X + \frac{1}{2}a^3Y - \frac{1}{2}a^2bH, \quad \eta = 2ad^2X - c^2Y + acdH,$$
$$\text{and} \quad \theta = 4bdX - 2acY + (a^2d + bc)H.$$

We define a volume element Ω on Σ^3 as follows: set

$$\Omega_X = d\delta_b - b\delta_d, \quad \Omega_Y = \frac{1}{2}(a\delta_c - 2c\delta_a), \quad \text{and} \quad \Omega_H = \frac{1}{2}(2ad\delta_a + c\delta_b - b\delta_c - a^2\delta_d),$$

and define $\Omega = \Omega_X \wedge \Omega_Y \wedge \Omega_H$. Here, $\delta_a(\partial_x) = 0$ if $x = b, c, d$ and 1 if $x = a$, and similarly for δ_b, δ_c and δ_d. One can easily compute the following

$$[H,X] = 4X, \quad [H,Y] = -3Y, \quad [X,Y] = aH,$$
$$div(X) = div(Y) = 0.$$

It follows that $div(aH) = 0$, and hence that $div(H) = 1$. Thus, since aH vanishes when $a = 0$, we need more than just X, Y and H to prove that Σ^3 is EMV. However, this is indeed the case.

Proposition 7.1. *Σ^3 is EMV.*

Proof. It suffices to show that H can be written as a sum $\sum f_jV_j$ with the V_j generated by X and Y. To this end,

$$\begin{aligned} ad[X,Y] + c[Y,[Y,X]] - 3acY &= a^2dH - c[Y,aH] - 3acY \\ &= a^2dH - c((Ya)H - a[Y,H]) - 3acY \\ &= (a^2d - bc)H + 3acY - 3acY \\ &= H. \end{aligned}$$

□

Moreover, we have been able to prove (with considerable difficulty) that if Σ^3 has the volume density property, then it has the density property. Nevertheless, the combinatorics arising in attempts to prove the volume density property by the methods of sections 4 and 5 become too cumbersome for us to handle.

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