EXTENSION OF JETS WITH L^2 ESTIMATES

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ABSTRACT. The problem of L^2 extension of normal jets from a hypersurface is studied, with focus on the growth order of the constant. Using aspects of the standard, twisted approach for L^2 extension and of the new approach to L^2 extension introduced by Berndtsson and Lempert, an extension theorem for normal k-jets is proved, with the norm of the extension operator bounded by a constant C^k, where C is universal. The jet extension theorem is then used to extend positively curved singular Hermitian metrics from smooth, deformably pseudoeffective hypersurfaces in projective manifolds.

INTRODUCTION

The extension of holomorphic functions and sections of line bundles with universal L^2 estimates is one of the most fundamental results in analysis and geometry of several complex variables. The first results regarding L^2 extension go back to Hörmander's celebrated Acta paper [H-1965, Theorem 2.5.1], where rather strong positivity and local boundedness conditions were imposed on the weights defining the L^2 norm. The most important breakthrough was achieved by Ohsawa and Takegoshi [OT-1987]. Since then, many authors have established the L^2 extension theorem to a still growing collection of analytic and geometric contexts, and many important applications have been found in the classical theory of several complex variables, complex analytic geometry, and birational algebraic geometry.

The present article establishes an L^2 extension theorem for normal jets to a smooth complex hypersurface in a Stein manifold. To state the result, let (X, ω) be a Stein Kähler manifold of complex dimension n, $Z \subset X$ a closed hypersurface, and $L \to X$ a holomorphic line bundle with singular Hermitian metric $e^{-\varphi}$. The hypersurface Z defines a holomorphic line bundle $L_Z \to X$ and there is a holomorphic section $T \in \Gamma_{\mathcal{O}}(X, L_Z)$ such that $Z = \{T = 0\}$. For $k \in \mathbb{N}$, L-valued normal k-jets on Z are sections of the quotient sheaf $\mathcal{O}_X(L)/\mathscr{I}_Z^{k+1}(L)$, where $\mathscr{I}_Z$ is the zero ideal sheaf of Z, so that $f \in \mathscr{I}_Z^m \iff f/T^m \in \mathcal{O}_X$ (c.f. Section 2). Write

$$\mathcal{J}^k_{\perp}(Z|X, L) := \mathcal{O}_X(L)/\mathscr{I}_Z^{k+1}(L) \quad \text{and} \quad J^k_{\perp} : \mathcal{O}_X(L) \to \mathcal{J}^k_{\perp}(Z|X, L).$$

As will be further detailed in Paragraph 2.2, the global sections of $\mathcal{J}^k_{\perp}(Z|X, L)$ correspond to $(k+1)$-tuples

$$(f_o, ..., f_k) \in \bigoplus_{\ell=o}^{k} \Gamma_{\mathcal{O}}(Z, L \otimes L_Z^{*\otimes \ell}).$$

The correspondence between $\gamma \in H^0(Z, \mathcal{J}^k_{\perp}(Z|X, L))$ and $(f_o, ..., f_k)$ will sometimes be written

$$\Pi^{(\ell)}_{\perp}\gamma := f_\ell, \quad o \le \ell \le k.$$

For metrics $e^{-\varphi}$ and $e^{-\lambda}$ for L and L_Z, define the Hilbert spaces

$$\mathfrak{H}^2_k := \left\{ \gamma \in H^0(Z, \mathcal{J}^k_{\perp}(Z|X, L)) \; ; \; ||\gamma||^2_k := \sum_{\ell=0}^{k} \int_Z \frac{|\Pi^{\ell}_{\perp}\gamma|^2 e^{-\varphi + \ell\lambda}}{|dT|^2_\omega e^{-\lambda}} dA_\omega < +\infty \right\}$$

and

$$\mathscr{H}^2 := \left\{ F \in \Gamma_{\mathcal{O}}(X, L) \ ; \ ||F||^2_X := \int_X |F|^2 e^{-\varphi} dV_\omega < +\infty \right\},$$

where $dA_\omega = \omega^{n-1}/(n-1)!$ and $dV_\omega = \omega^n/n!$. The first main result is the following theorem.

THEOREM 1. *Let (X, ω) be a Stein Kähler manifold and let $Z \subset X$ be a smooth complex hypersurface. Let $T \in H^0(X, L_Z)$ be a section cutting out Z, and $e^{-\lambda}$ a singular metric for L_Z, such that*

$$\partial\bar\partial\lambda \geq 0, \quad e^{-\lambda}|_Z \not\equiv -\infty \quad \textit{and} \quad \sup_X |T|^2 e^{-\lambda} \leq 1. \tag{1}$$

Let $\delta > 0$ and let $L \to X$ be a holomorphic line bundle with singular Hermitian metric $e^{-\varphi}$ such that

$$(\partial\bar\partial\varphi + \mathrm{Ricci}(\omega)) \geq (k + 1 + \delta)\partial\bar\partial\lambda.$$

Then there is a universal constant C such that for each $\gamma \in \mathfrak{H}^2_k$ there exists $F \in \mathscr{H}^2$ satisfying

$$J^k_\perp F = \gamma \quad \textit{and} \quad ||F||^2 \leq \frac{k+1+\delta}{\delta} C^k ||\gamma||^2_k.$$

The proof of Theorem 1 begins with the extension of normal k-jets to a small neighborhood of the hypersurface Z. The authors developed the technique for carrying out this part of the proof some time ago, but the technique relies heavily on the neighborhood being arbitrarily small, and for a long time there was no obvious way to extend the methods to a general domain.

Things changed with the appearance of the work of Berndtsson and Lempert [BL-2016], who gave a new proof of the L^2 extension theorem in a special case. The thrust of the work of Berndtsson and Lempert is a degeneration technique, which degenerates the ambient manifold onto the hypersurface from which L^2 extension is to be carried out. The result Berndtsson and Lempert prove is that certain quantities increase with this type of degeneration. One of these increasing quantities is the norm of the operator sending a section on the hypersurface to its extension of minimal L^2 norm.

The degeneration technique of Berndtsson and Lempert relies on the celebrated work [B-2009] of Berndtsson regarding plurisubharmonic variation of Hilbert spaces. Although it is well-known by now, a short and more direct proof of a (slightly) special case of Berndtsson's Theorem is provided in Section 3.

The hypersurfaces considered in Theorem 1 are not completely general. Rather, they have a property that can be described as follows. Given any complex hypersurface $Z \subset X$, the line bundle $L_Z \to X$ has a metric $e^{-\lambda}$ with non-negative curvature satisfying $\sup_X |T|^2 e^{-\lambda} \leq 1$, namely $\lambda = \log |T|^2$, but of course this metric is singular on Z. Thus the kind of hypersurfaces considered in Theorem 1 are those for which *the singular locus of the metric of minimal singularities for L_Z does not have any component whose support is a component of Z*. Such hypersurfaces are not just pseudoeffective, but in an infinitesimal sense they move in a pseudoeffective family.

REMARK 1.1. Recall that Demailly's metric of minimal singularities for a pseudoeffective holomorphic line bundle is the minimal element of the set of equivalence classes of all singular Hermitian metrics with respect to the following order relation:

$$e^{-\varphi_1} \prec e^{-\varphi_2} \iff e^{-\varphi_1} \leq C e^{-\varphi_2} \text{ for some positive continuous function } C : X \to (0, \infty).$$

The equivalence classes are defined via this order relation: two metrics $e^{-\varphi_1}$ and $e^{-\varphi_2}$ to be equivalent if and only if $e^{-\varphi_1} \prec e^{-\varphi_2}$ and $e^{-\varphi_2} \prec e^{-\varphi_1}$. ◇

DEFINITION 1.2. A hypersurface Z is said to be *deformably pseudoeffective* if the metric of minimal singularities of L_Z is locally bounded at some point of every irreducible component of Z.

The second main result of the paper can now be stated.

THEOREM 2. *Let X be a smooth complex projective manifold and let $Z \subset X$ be a smooth, deformably pseudoeffective complex hypersurface. Let $H \to X$ be a pseudoeffective line bundle and fix a singular Hermitian metric $e^{-\varphi_o}$ for H such that $\sqrt{-1}\partial\bar{\partial}\varphi_o \ge 0$ as a current. Then for any singular Hermitian $e^{-\varphi}$ for $H|_Z \to Z$ such that*

$$\sqrt{-1}\partial\bar{\partial}\varphi \ge 0 \quad \textit{and} \quad \frac{e^{-\varphi}}{e^{-\varphi_o}} \ge 1 \textit{ on } Z$$

there exists a singular Hermitian metric $e^{-\Phi}$ for $H \to X$ such that

$$\sqrt{-1}\partial\bar{\partial}\Phi \ge 0 \quad \textit{and} \quad e^{-\Phi}|_Z = e^{-\varphi}.$$

On a compact complex manifold, a pseudoeffective line bundle admits, by definition, a singular Hermitian metric whose curvature is a non-negative $(1,1)$-current. The name comes from Algebraic Geometry, where a pseudoeffective line bundle L is a line bundle lying in the closure of the effective cone, i.e., for any ample line bundle A and any positive integer m, there exists a positive integer k such that $k(mL + A)$ has a non-identically zero section.

When the line bundle $H \to X$ is ample, Theorem 2 is due to Coman, Guedj and Zeriahi [CGZ-2010]; in fact, they establish the result for all submanifolds of X, with no additional restrictions. (Collins and Tosatti [CT-2013] give another proof in the ample case, among a number of other results.) By the well-known theorem of Kodaira, the ampleness of H is equivalent to the existence of a smooth metric of strictly positive curvature. In this case, the conditions of Theorem 2 are satisfied for *any* singular Hermitian metric, and thus Theorem 2 is a generalization of the result of [CGZ-2010] for deformably effective hypersurfaces.

It is known [M-2013] that if there are no conditions placed on the singularities of the metric to be extended, and if extension is possible from *any* subvariety, then either $H \to X$ is ample or $c_1(H) = 0$.

The proof of Theorem 2 follows ideas communicated to us by Guedj almost six years ago, when one of us visited Toulouse. According to Guedj, this was the initial approach that he, Coman and Zeriahi tried to use for the proof of Theorem 2 in the ample case, but the results on the L^2 extension of jets available at that time were not sufficiently sharp.

Theorem 2 is confined to the setting of hypersurfaces that are deformably pseudoeffective. The *hypersurface* aspect is not important; the proof generalizes to submanifolds of higher codimension, but the authors have opted to restrict attention to the setting of hypersurfaces for reasons of personal preference. On the other hand, the assumption that Z is deformably pseudoeffective is necessary for the proof, primarily because the method of Berndtsson and Lempert (particularly, the ideas of Section 5) has not yet been adapted to the setting of general hypersurfaces.

CONJECTURE. Theorem 2 holds even if the words *deformably pseudoeffective* are removed.

Theorem 1 is not the first result on the L^2 extension of jets. To the best of the authors' knowledge, the first such result was proved by Popovici [P-2005], with two differences, the second of which is very important in the present paper, especially for the proof of Theorem 2: in Popovici's Theorem (i) the constant C is not universal, but rather depends on the metric $e^{-\lambda}$ (which also has to be smooth), and (ii) the constant appearing is of the form C^{k^2}, and not C^k.

More recently, Demailly [D-2015] has considered L^2 extension of holomorphic functions from non reduced analytic subschemes, and this work was extended by Cao, Demailly and Matsumura [CDM-2017]. The flavor of the results in these works is rather different from that of the present article.

One of the remarkable aspects of the work [BL-2016] of Berndtsson and Lempert is that it yields a sharp bound on the norm of the operator that assigns to a given section its extension of minimal norm. The so-called *sharp constants* result was already proved, first by Z. Błocki [B-2013] in a special case of domains, and then by Q. Guan and X. Zhou [GZ-2015] in general. Their proofs, while not identical, are very close in spirit (and the fundamental new idea in both proofs is the same), and essentially refine, in a non-trivial way, the standard approach to L^2 extension. By contrast, the proof of Berndtsson and Lempert is radically different, relying as we said on Berndtsson's Theory of (pluri)subharmonic variation of Hilbert spaces. It is interesting to note, however, that the extension theorem with sharp constants is *equivalent* to the theorem of Berndtsson on plurisubharmonic variation. This was partially seen in [GZ-2015], and further discussed in the work [HPS-2016] of Hacon, Popa and Schnell, but a complete proof has not yet been written.

The present article does not consider the question of sharp constants for jet extension, since for the application, i.e., Theorem 2, only the universality of the constant plays a role. For the reader interested in the sharp constants for jets, the non-sharpness manifests in the first part of the proof of Theorem 1, where extension to a small neighborhood of the hypersurface is proved (c.f. Theorem 4.6). In fact, we are quite optimistic that the method of Błocki-Guan-Zhou can be adapted to sharpen the constant at that stage, and then it is clear from the rest of the proof that one would obtain a sharp constant. However, this sharpening would have added a significant amount of additional detail to the paper, and thus was abandoned in favor of a simpler approach that is sufficient for the proof of Theorem 2 and many other applications.

While this article was being written, the preprint [Hos-2017] appeared on the ArXiV database. In Hosono's article the approach of Berndtsson and Lempert is used to prove L^2 extension of jets in the flat case. (Certainly Hosono can extend his results to the case of deformably pseudoeffective hypersurfaces without a lot of extra effort. However, the case of jet extension from a general subvariety is still unproved.) The jet norms of Hosono are quite different form ours, and are quite similar to the sorts of results obtained at the end of the article [B-2017]. We do not know how to use Hosono's result to give a proof of Theorem 2.

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CONTENTS

2. Background and Notation

2.1. **Homogeneous expansion.** The proof of Theorem 1 requires a lifting of the extension problem to the disk bundle

$$\mathscr{B}(\lambda) := \{v \in L_Z^* \ ; \ |v|^2 e^{\lambda} < 1\}.$$

As is well-known, the vertical boundary

$$\partial_V \mathscr{B}(\lambda) := \{v \in L_Z^* \ ; \ |v|^2 e^{\lambda} = 1\}$$

is pseudoconvex (with respect to $\mathscr{B}(\lambda)$) if and only if $e^{-\lambda}$ has non-negatve curvature, which is the case for us. Consequently, since the base X is Stein, so is the disk bundle $\mathscr{B}(\lambda)$.

Proposition 2.1. *Let $L, H \to X$ be holomorphic line bundles and denote by $\pi : H^* \to X$ the dual bundle. Let $\sigma \in \Gamma_{\mathcal{O}}(H^*, \pi^* H^*)$ denote the diagonal section*

$$\sigma(v) := (v, v).$$

Then for any $s \in \Gamma_{\mathcal{O}}(H^*, \pi^* L)$ there exist sections $a_j \in \Gamma_{\mathcal{O}}(X, H^{\otimes j} \otimes L)$, $j = 0, 1, \ldots$, such that

$$s = \sum_{j=0}^{\infty} (\pi^* a_j) \otimes \sigma^{\otimes j}.$$

Proof. Fix some $v \in H^*$ and a frame η for $L \to X$ near $x = \pi v$. Let $\xi \in H_v^* - \{0\}$ and let $t\xi$ be a typical point on the fiber H_v. Writing our section as $s = f \pi^* \eta$, we have the power series expansion

$$f(t\xi) = \sum_{j \geq 0} A_j(\xi) t^j.$$

If one starts with another $\xi' \in H_v^* - \{0\}$ and $t' \in \mathbb{C}$ such that $t\xi = t'\xi'$ then

$$\sum_{j \geq 0} A_j(\xi) t^j = f(t\xi) = f(t'\xi') = \sum_{j \geq 0} A'_j(\xi') t'^j,$$

from which it follows that

$$a_j(\pi\xi) := A_j(\xi) \eta \otimes \xi^{\otimes -j}$$

is well-defined, independent of ξ. Thus $a_j \in \Gamma_{\mathcal{O}}(X, L \otimes H^{\otimes j})$, and

$$\sum_{j=0}^{\infty} \left(\pi^* a_j \otimes \sigma^{\otimes j}\right)(v) = \sum_{j=0}^{\infty} A_j(v)(v)^{\otimes -j} \sigma(v)^{\otimes j} \otimes \pi^*\eta = \sum_{j=0}^{\infty} A_j(v)\pi^*\eta = f\pi^*\eta,$$

as required. □

2.2. **Normal jets.** Let X be a complex manifold and $S \subset X$ a complex submanifold of complex codimension r. Fix a holomorphic line bundle $H \to X$.

2.2.1. Normal k-jets. Given an ideal $\mathscr{I} \subset \mathcal{O}_X$, one says that two germs in $\mathcal{O}_{X,x}$ are tangent to order $k+1$ along the normal directions to $\mathscr{I}$ if their difference lies in $\mathscr{I}_x^{k+1}$. When $\mathscr{I} = \mathscr{I}_S$ is the sheaf of germs of holomorphic functions that vanish along a complex submanifold S, tangency to order $k+1$ means that if we choose a coordinate system so that S is locally a factor in a Cartesian product, then when we expand the germs in a Taylor series in the complementary variables to S (so the coefficients of the Taylor series live in $\mathcal{O}_S$), the two series coincide up to order k. Since the definitions are local, they make sense if we also twist by a holomorphic vector bundle.

DEFINITION 2.2. Fix a holomorphic line bundle $H \to X$ and let $k \in \mathbb{N}_+$.

(1) An H-valued normal k-jet to S at $x \in S$ is an equivalence class of germs of holomorphic sections of $H \to X$ at x that are tangent to order $k+1$ along the normal directions to S, i.e., an element of the sheaf $\mathcal{J}_\perp^k(S|X, H) := (\mathcal{O}_X/\mathscr{I}_S^{k+1})(H)$.
(2) The normal k-jet determined by the germ of a section $F \in \mathcal{O}_X(H)$ is denoted $J_\perp^k F$. ⋄

One has the exact sequence of sheaves

$$0 \to \mathcal{O}_S(H \otimes \mathrm{Sym}^k(N_{X/S}^*)) \to \mathcal{J}_\perp^k(S|X, H) \to \mathcal{J}_\perp^{k-1}(S|X, H) \to 0. \tag{2}$$

The map sending a section $P \in \Gamma_{\mathcal{O}}(S \cap U, H \otimes \mathrm{Sym}^k N_{X/S}^*)$ of the H-twisted k^{th} symmetric power of the co-normal bundle of S in X to an H-valued k-jet can be described as follows: if we take normal coordinates $z^1, ..., z^r$ to S, then $dz^1|_S, ..., dz^r|_S$ is a frame for $N_{X/S}^*$ over U then we can write $P(s) = f_\alpha(s) dz^\alpha \otimes \xi$. One can then consider $F_P := f_\alpha(s) z^\alpha$ and compute $J_\perp^k F_P$. Different choices of normal coordinates will result in the same term of order k, but the lower order terms can change.

By computing Taylor expansions in the coordinates $z^1, ..., z^r$ one can see that the k-th order term of any k-jet is realized by a section of the H-twisted k^{th} symmetric power of the co-normal bundle as in the previous paragraph.

In general, the sequence (2) does not split globally holomorphically. Nevertheless, for each $F \in \Gamma_{\mathcal{O}}(X, H)$ we can define a sequence of holomorphic sections

$$f_k \in \Gamma_{\mathcal{O}}(S, H \otimes \mathrm{Sym}^k(N_{X/S}^*))$$

by letting $f_o := F|_Z$, and taking f_k to be the unique element that maps to $J_\perp^k F$ in (2). Note that the section f_k depends on F only implicitly through $J_\perp^k F$, i.e., if $J_\perp^k F = J_\perp^k \tilde{F}$ then F and $\tilde{F}$ yield the same section f_k.

DEFINITION 2.3. The section f_k is called the k^{th} Taylor coefficient of $F \in \Gamma_{\mathcal{O}}(X, L)$ relative to S. When it is convenient, the notation

$$f_j := F_\perp^{(j)} \quad \text{or} \quad f_j := \Pi_\perp^{(j)} J_\perp^k F,$$

the latter for $j \leq k$, might also be used.

Definition 2.3 can be misleading; in general one cannot form a Taylor series, or even a Taylor polynomial, on X (or even on a neighborhood of S) from the Taylor coefficients. (See the discussion after Definition 2.4.)

2.2.2. Normal k-jets to a hypersurface Z in a Stein manifold. Let X be a Complex manifold and $Z \subset X$ a smooth hypersurface. Let T be the defining section for the holomorphic line bundle L_Z associated to Z. The smoothness of Z implies that the section dT of $N^*_{X/Z} \otimes L_Z|_Z$, which is well-defined on the zero set of T, is nowhere-zero, and hence the latter line bundle is trivial. Therefore $L_Z|_Z$ is isomorphic to the normal bundle $N_{X/Z}$ of Z in X. In particular, L_Z gives an extension of the normal bundle of Z to all of X. This observation is often referred to as *adjunction*.

As above, to a normal k-jet section $\gamma = J^k_\perp F$ of a section $F \in \Gamma_{\mathcal{O}}(X, H)$ one associates the $(k+1)$-tuple

$$(f_o, ..., f_k) \in \bigoplus_{j=0}^{k} \Gamma_{\mathcal{O}}(Z, H \otimes L_Z^{*\otimes j})$$

of Taylor coefficients relative to Z, i.e., $f_j = \Pi^{(j)}_\perp \gamma$, $0 \le j \le k$. The L^2 norm of a normal k-jet $\gamma = J^k_\perp F$ is then defined in terms of the sections f_j as follows.

DEFINITION 2.4. Let $H \to X$ be a holomorphic line bundle, ω be a Hermitian Riemannian metric on X, $e^{-\varphi}$ a Hermitian metric for $H \to X$, and $e^{-\lambda}$ a Hermitian metric for $L_Z \to X$. The square of the L^2-norm of the k-jet section $\gamma \in H^0(Z, \mathcal{J}^k_\perp(Z|X, H))$ is

$$||\gamma||^2_k := \sum_{\ell=0}^{k} \int_Z \frac{|\Pi^{(\ell)}_\perp \gamma|^2 e^{-(\varphi - \ell\lambda)}}{|dT|^2_\omega e^{-\lambda}} dA_\omega.$$

In trying to tie the $(k+1)$-tuple $(f_o, ..., f_k)$ of sections on Z to a section $J^k_\perp F$ on Z obtained from some section F on X, one has in mind that there is an expansion

$$F = f_o + Tf_1 + ... + T^{\otimes k} f_k + O(T^{\otimes (k+1)}).$$

Of course, if this expression is taken literally then it does not capture the jet; it is only defined on Z, where it equals f_o. To get F to be a section on X, the f_j must be extended to sections $\tilde{f}_j$ on X, but even after carrying out this extension, the $k+1$-tuple associated to F need not be $(f_o, ..., f_k)$. However, if the extensions $\tilde{f}_j$ of the f_j osculate f_j to order $k+1-j$ along Z then $(f_o, ..., f_k)$ is recovered from F.

In a general complex manifold it is not possible to extend the sections f_i, much less extend them in the right way. However, on a Stein manifold, such extensions are possible, as the following proposition shows.

PROPOSITION 2.5. *Let X be a Stein manifold, $H \to X$ a holomorphic line bundle, and $Z \subset X$ a smooth complex hypersurface with defining section T. Let $f = (f_o, ..., f_k)$ be a $(k+1)$-tuple of sections with $f_j \in \Gamma_{\mathcal{O}}(Z, H \otimes L_Z^{*\otimes j})$. Then there exist sections $\tilde{f}_j \in \Gamma_{\mathcal{O}}(X, H \otimes L_Z^{*\otimes j})$, $0 \le j \le k$, such that, with*

$$F := \tilde{f}_o + \tilde{f}_1 \otimes T + \cdots + \tilde{f}_k \otimes T^{\otimes k} \in H^0(X, H), \tag{3}$$

$F^{(j)}_\perp = f_j$ *for every $j = 0, 1, ..., k$. In particular, there is a section $F \in \Gamma_{\mathcal{O}}(X, H)$ whose $(k+1)$-tuple of Taylor coefficients is $(f_o, ..., f_k)$.*

Proof. Let $j \in \{o, 1, ..., k\}$.

Choose open sets $V_\mu \subset\subset U_\mu \subset\subset X$, $\mu \ge 1$, such that $Z \subset \bigcup_{\mu \ge 1} V_j$ and each U_μ is a product neighborhood $W_\mu \times D_\mu$ with coordinates $(z_\mu, t_\mu) \in \mathbb{C}^{n-1} \times \mathbb{C}$ such that $W_\mu = \{t_\mu = 0\}$ is biholomorphic to the unit ball, $\{W_\mu\}$ is an open cover of Z and $D_\mu = \{|t_\mu| < 1\}$. Note that every holomorphic line bundle on U_μ is trivial. Let $U_o := X - \bigcup_{\mu \ge 1} V_\mu$, so that $\{U_\mu\ ;\ j \ge o\}$ is an open cover of X. Choose frames η_μ for $L_Z|_{U_\mu}$ such that $T|_{U_\mu} = t_\mu \eta_\mu$. Fix frames ξ_μ for $H \otimes L_X^{*\otimes j}|_{U_\mu}$. Finally, fix a partition of unity $\{\phi_\mu\}$ subordinate to $\{W_\mu\}$ and a function $\sigma \in \mathscr{C}^\infty_o(\{|t| < 1\})$ such that $\sigma|_{|t| \le 1/2} \equiv 1$, and define

$$\chi_\mu(z_\mu, t_\mu) := \phi(z_\mu)\sigma(t_\mu), \quad \mu \ge 1.$$

Fix any $\chi_o \in \mathscr{C}^\infty_o(U_o)$.

Let $g_j^o := 0 \in \Gamma_{\mathcal{O}}(U_o, H \otimes L_Z^{*\otimes j})$. For $\mu \ge 1$, write $f_j|_{U_\mu} =: h_j^\mu(z_\mu)\xi_\mu(z_\mu)$ and define the local extension

$$g_j^\mu(z_\mu, t_\mu) := h_j^\mu(z_\mu)\xi_\mu(z_\mu, t_\mu).$$

Define $\tilde{g}_j := \sum_\mu \chi_\mu g_j^\mu$. Note that $\tilde{g}_j$ is a smooth section of $H \otimes L_Z^{*\otimes j} \to X$ that is holomorphic on a neighborhood of Z in X, satisfies $\tilde{g}_j|_Z = f_j$, and locally osculates f_j to infinite order near Z. (In fact, $\tilde{g}_j$ agrees with the constant extension of f_j along the normal directions fixed by the choice of product neighborhood $W_\mu \times D_\mu$.) Let $\alpha_j := \bar\partial \tilde{g}_j$.

Fix a Kähler form ω on X and metrics $e^{-\kappa}$ and $e^{-\tau}$ for H and L_Z respectively, and choose a sufficiently rapidly growing strictly plurisubharmonic exhaustion ψ for X such that

$$\sqrt{-1}\partial\bar\partial(\psi + \kappa + \mathrm{Ricci}(\omega) - (k + j + 1)\tau) \ge \omega \quad \text{and} \quad \int_X \frac{|\alpha_j|^2_\omega e^{-\kappa + (j+k+1)\tau + \psi}}{|T|^{2k+2}} dV_\omega < +\infty.$$

By Hörmander's Theorem there exists a section u_j of $H \otimes L_Z^{*\otimes j} \to X$ such that

$$\bar\partial u_j = \alpha_j \quad \text{and} \quad \int_X \frac{|u_j|^2 e^{-\kappa + (j+k+1)\tau + \psi}}{|T|^{2k+2}} dV_\omega < +\infty.$$

In particular, since α_j is smooth, so is u_j, and thus $u_j = O(T^{\otimes(k+1)})$ near Z.

Let $\tilde{f}_j := \tilde{g}_j - u_j$. Then the first $k + 1$ Taylor coefficients of the section $F \in \Gamma_{\mathcal{O}}(X, H)$ defined by (3) are clearly $(f_o, ..., f_k)$. □

REMARK 2.6. In case the hypersurface Z is not smooth, one still has an extension of jets by general Stein theory, but control of the splitting into Taylor coefficients is lost near the singular locus. Such a lift will be used implicitly in Section 4, but we shall work with this extension near the smooth points. ◇

2.3. $\mathbf{L^2}$ Extension of sections.

The following results regarding L^2 extension will be used in the proof of Theorem 2.

2.3.1. L^2 extension of sections from a hypersurface.

In the special case $k = 0$ Theorem 1 has been known for quite some time. For example, in [V-2008] the following result was proved.

THEOREM 2.7. *Let (X, ω) be a Stein Kähler manifold and let $Z \subset X$ be a smooth complex hypersurface. Let $T \in \Gamma_{\mathcal{O}}(X, L_Z)$ be a section cutting out Z, and $e^{-\lambda}$ a singular Hermitian metric for L_Z, such that*

$$\sup_X |T|^2 e^{-\lambda} \le 1.$$

Fix $\delta \in (0,1]$. Let $L \to X$ be a holomorphic line bundle with singular Hermitian metric $e^{-\varphi}$ such that

$$\sqrt{-1}(\partial\bar{\partial}\varphi + \mathrm{Ricci}(\omega)) \geq (1+\alpha\delta)\sqrt{-1}\partial\bar{\partial}\lambda \qquad \textit{for all } \alpha \in [0,1].$$

Then for each $f \in \Gamma_{\mathcal{O}}(Z, L)$ such that

$$\int_Z \frac{|f|^2 e^{-\varphi}}{|dT|^2_\omega e^{-\lambda}} dA_\omega < +\infty$$

there exists $F \in \Gamma_{\mathcal{O}}(X, L)$ such that

$$F|_Z = f \quad \textit{and} \quad \int_X |F|^2 e^{-\varphi} dV_\omega \leq \frac{24\pi}{\delta} \int_Z \frac{|f|^2 e^{-\varphi}}{|dT|^2_\omega e^{-\lambda}} dA_\omega.$$

Of course, Theorem 1 is a generalization of Theorem 2.7 if one doesn't care about the constant $24\pi/\delta$ (and in fact, this constant is not needed in the present article; only its universality is important). In the spirit of economy and self containment, we note that our proof of Theorem 1 is self-contained (though, as the reader will find, somewhat different from that of [V-2008]).

2.3.2. L^2 extension of sections from a point.

PROPOSITION 2.8. *Let X be a projective manifold of complex dimension n. Let ω be the Kähler metric associated to the curvature of the Fubini-Study metric of some projective embedding of X. Let $H \to X$ be a holomorphic line bundle with singular Hermitian metric $e^{-\kappa}$ whose curvature current $\sqrt{-1}\partial\bar{\partial}\kappa$ satisfies*

$$\sqrt{-1}\partial\bar{\partial}\kappa + \sqrt{-1}\mathrm{Ricci}(\omega) \geq (n+\varepsilon)\omega \tag{4}$$

for some positive number ε. Then there is a universal constant $C > 0$ depending only on ε and the dimension n, and not the line bundle H or the metric $e^{-\kappa}$, with the following property. For each $x \in X$ and each $a \in H_x$ such that $|a|^2 e^{-\kappa(x)} < +\infty$ (with the convention that $0 \times \infty = 0$) there exists $f \in H^0(X, H)$ such that

$$f(x) = a \quad \textit{and} \quad \int_X |f|^2 e^{-\kappa} dV_\omega \leq C|a|^2 e^{-\kappa(x)}.$$

Sketch of proof. Let $A \to X$ be the very ample line bundle associated to a hyperplane section of the projective embedding of X, so that A has a smooth metric $e^{-\chi}$ whose curvature is ω. Choose holomorphic sections $s_1, ..., s_n \in H^0(X, A)$ whose zero sets are smooth divisors that meet pairwise transversely, such that

$$\{x\} = \bigcap_{j=1}^{n} \{s_j = 0\}.$$

Apply the extension theorem 2.7 inductively to the smooth sub varieties

$$\{x\} = H_0 \subset H_1 \subset \cdots \subset H_{n-1} \subset H_n = X,$$

where $H_i = \bigcap_{j=1}^{n-i} \{s_j = 0\}$, and the metrics $e^{-\varphi} = e^{-\kappa}$ and $e^{-\lambda_Z} = e^{-(n-i)\chi}$. The details are left to the reader. □

2.4. **Approximation of singular Hermitian metrics on ample line bundles.** A technique introduced by Demailly can be used to approximate metrics of Hermitian line bundles by so-called *algebraic metrics*. Demailly's results apply to quite general line bundles. However, in the ample case, there is a simplification that shortens proofs. The result is stated here, since it will be needed in the proof of Theorem 2. Since the proof is rather straightforward in this special case, we present it here as well for the reader's convenience.

LEMMA 2.9. *Let X be a smooth projective variety, let $H \to X$ a pseudoeffective line bundle with a singular Hermitian metric $e^{-\varphi}$ having non-negative curvature current, and let A be an ample line bundle with smooth Hermitian metric $e^{-\chi}$ of positive curvature. Fix a Kähler form ω on X. Then for all $m \ge 0$, if the holomorphic sections $h_1^{(m)}, ..., h_{N_m}^{(m)} \in H^0(X, H^{\otimes m} \otimes A)$ form an orthonormal basis for the Hilbert space*

$$\mathscr{H}^2_m(\varphi) := \left\{ f \in H^0(X, H^{\otimes m} \otimes A) \ ; \ ||f||^2_m := \int_X |f|^2 e^{-(m\varphi + \chi)} dV_\omega < +\infty \right\}$$

then the metric $e^{-\psi_m}$ for $H \to X$, defined by the potential

$$\psi_m := -\frac{\chi}{m} + \frac{1}{m} \log \sum_{j=1}^{k} |h_j^{(m)}|^2$$

satisfies

$$(Cr^{-2n})^{1/m} e^{\frac{\chi(x)}{m}} \inf_{B_\omega(x,r)} e^{-(\varphi + \frac{1}{m}\chi)} \le e^{-\psi_m(x)} \le C^{1/m} e^{-\varphi(x)} \tag{5}$$

for some constants $C > 0$ and $0 < r << 1$ independent of m and x. ◇

REMARK 2.10. Strictly speaking, the metric $e^{-\varphi}$ is not a function, so the expression on the left hand side of (5) does not make sense. There are a couple of ways to make sense of it. One way is to fix a set of local frames for H and A on a cover of X such that on each open set of the cover, H and A are trivial. Measuring the length of each frame, with respect to $e^{-\varphi}$ or $e^{-\chi}$ as needed, yields functions. Since X is compact, we can choose the cover to be finite.

An almost equivalent approach is to fix once and for all smooth metric $e^{-\lambda}$ and $e^{-\mu}$ for H and A. Then one proves the inequality

$$\frac{C^{1/m}}{r^{2n/m}} e^{\frac{\chi(x)-\mu(x)}{m}} \inf_{B_\omega(x,r)} e^{-(\varphi - \lambda + \frac{1}{m}(\chi - \mu))} \le e^{\lambda(x) - \psi_m(x)} \le C^{1/m} e^{\lambda(x) - \varphi(x)},$$

and again the constants C and r are independent of m and x. ◇

Proof of Lemma 2.9. A simple and well-known argument from the theory of Bergman kernels shows that

$$e^{m\psi_m(x) - m\varphi(x)} = \sup \left\{ |f(x)|^2 e^{-(m\varphi(x) + \chi(x))} \ ; \ ||f||_m = 1 \right\}.$$

We begin by establishing the estimate $e^{-\psi_m} \le C^{1/m} e^{-\varphi}$. Observe that the inequality holds for any point $x \in X$ such that $e^{-\varphi(x)} = \infty$, so we may assume that x is not a pole of φ. Assuming χ is chosen to be sufficiently positively curved, Proposition 2.8 implies that for each $a \in H^{\otimes m} \otimes A$ there exists $f \in \mathscr{H}^2_m$ such that

$$f(x) = a \quad \text{and} \quad \int_X |f|^2 e^{-((m\varphi + \chi)} dV_\omega \le C|a|^2 e^{-((m\varphi(x) + \chi(x))}.$$

Choosing a so that $C|a|^2 e^{-((m\varphi(x)+\chi(x))} = 1$, we see that

$$\sup\left\{|f(x)|^2 e^{-(m\varphi(x)+\chi(x))}\ ;\ ||f||_m = 1\right\} \ge C^{-1},$$

which establishes the desired upper bound.

To establish the lower bound, one proceeds as follows. Once again, if x is a pole for φ all of the sections in $\mathscr{H}^2_m$ must vanish at x, and lower bound trivially holds. Assuming x is not a pole of φ, choose a frame for $mH + A$ near x such that $e^{-((m\varphi(x)+\chi(x))} = 1$, and continue to write f for the local form of a given section of norm 1. We also work in a local coordinate z such that $z(x) = 0$. By the sub-mean value property and Jensen's Inequality, one has

$$\begin{aligned}\log|f(x)|^2 &\le \fint_{|z|\le\delta} \log\left(e^{m\varphi+\chi}|f|^2 e^{-(m\varphi+\chi)}\right) dV \\ &\le \log\left(\fint_{|z|\le\delta} e^{m\varphi+\chi}|f|^2 e^{-(m\varphi+\chi)} dV\right) \\ &\le \sup_{|z|\le\delta}(m\varphi+\chi) + \log\frac{C}{\delta^{2n}}.\end{aligned}$$

Dividing by m and taking the supremum over all sections $f \in \mathscr{H}^2_m$ of unit norm, we have

$$\psi_m(x) \le -\frac{\chi(x)}{m} + \sup_{|z|\le\delta}(\varphi(z) + \tfrac{1}{m}\chi(z)) + \frac{\widetilde{C}}{m}.$$

Finally, choosing a small r so that $B_\omega(x, r)$ contains the ball $|z| \le \delta$, we have

$$e^{-\psi_m(x)} \ge (r^{-2n}C_o)^{1/m} e^{\frac{\chi(x)}{m}} \inf_{B_\omega(x,r)} e^{-(\varphi+\frac{1}{m}\chi)}.$$

The proof is therefore complete. □

3. Berndtsson's Theorem on Plurisubharmonic Variation

In his article [B-2009] Berndtsson introduced the method of plurisubharmonic variation of Bergman spaces. The basic result of Berndtsson, loosely speaking, is that if a family of plurisubharmonic weights, parametrized by a pseudoconvex domain, varies in a plurisubharmonic fashion then the associated Hilbert bundle over this domain, whose fibers are Hilbert spaces of square-integrable functions with respect to these weights, has positive curvature in the sense of Nakano.

As Berndtsson and Lempert showed, for the purpose of L^2 extension the above pseudoconvex domain can be the unit disk $\mathbb{D} \subset \mathbb{C}$, in which case Nakano positivity coincides with the weaker notion of Griffiths positivity. Since Berndtsson's Theorem was not proved for manifolds, but only for domains, we extend the result to the manifold setting for the sake of completeness. The idea is due to Berndtsson, and we claim no originality. We do, however, provide a slightly different proof than that of Berndtsson's; in fact our proof is closer in spirit to the proof of Theorem 1.2 of [B-2009] than to the proof of Theorem 1.1.

3.1. **The general case.** Let Y be a complex manifold of complex dimension n and let $L \to Y$ be a holomorphic line bundle. Denote by $\wp : Y \times \mathbb{D} \to Y$ the projection to the first factor, and let $e^{-\varphi}$ be a smooth Hermitian metric for $\wp^* L \to Y \times \mathbb{D}$. There is a natural isomorphism of line bundles

$$\iota_\tau : L \to \wp^* L|_{Y\times\{\tau\}},$$

and we write

$$e^{-\varphi_\tau} := \iota_\tau^* e^{-\varphi}$$

for the metric for $L \to Y$ induced from $e^{-\varphi}|_{Y\times\{\tau\}}$ by this identification. Fix a Hermitian metric ω on Y. Then one has L^2 structures for sections of $L \to Y$ defined by the norm

$$||f||_\tau^2 := \int_Y |f|^2 e^{-\varphi_\tau} \frac{\omega^n}{n!}.$$

The Hilbert spaces $\mathscr{H}^2_\tau$ of all square integrable holomorphic sections $f \in H^0(Y, L)$ are then the fibers of a Hilbert bundle $\mathscr{H}^2 \to \mathbb{D}$. If additional conditions on $e^{-\varphi}$ are assumed (for example, if Y is a bounded domain in some larger manifold Y', and if $e^{-\varphi}$ is the restriction of some smooth metric on $Y' \times \mathbb{C}$, say) then this bundle is a trivial vector bundle over $\mathbb{D}$, and in fact the vector subspaces $\mathscr{H}^2_\tau \subset \Gamma_{\mathcal{O}}(Y, L)$ are independent of τ. The Hilbert space structure of the fibers varies quasi-isometrically, i.e., there exist constants $C_{\sigma,\tau} > 0$, $\tau, \sigma \in \mathbb{D}$ such that

$$||f||_\sigma \le C_{\sigma,\tau} ||f||_\tau \quad \text{for all } f \in \mathscr{H}^2_\tau.$$

It follows that the dual spaces $\mathscr{H}^{2*}_\tau$ of bounded linear functionals with the dual norm

$$||\ell||_{\tau *} := \inf \left\{ C > 0 \; ; \; |\ell f| \le C ||f||_\tau \text{ for all } f \in \mathscr{H}^2_\tau \right\} = \sup_{f \in \mathscr{H}^2_\tau - \{0\}} \frac{|\ell f|}{||f||_\tau}$$

are also independent of τ as vector spaces, and vary quasi-isometrically as Hilbert spaces. Therefore there is a dual trivial bundle $\mathscr{H}^{2*} \to \mathbb{D}$.

DEFINITION 3.1 (Sections of the Hilbert bundles).

(i) A section of $\mathscr{H}^2 \to \mathbb{D}$ is a section s of $\wp^* L \to Y \times \mathbb{D}$ such that for each $\tau \in \mathbb{D}$ the section $s_\tau := \iota_\tau^* s$ of $L \to Y$ lies in $\mathscr{H}^2_\tau$. The collection of sections of $\mathscr{H}^2 \to \mathbb{D}$ is denoted $\Gamma(\mathbb{D}, \mathscr{H}^2)$.

(ii) We say that the section $s \in \Gamma(\mathbb{D}, \mathscr{H}^2)$ is holomorphic if $s \in H^0(Y \times \mathbb{D}, \wp^* L)$. The collection of holomorphic sections of $\mathscr{H}^2 \to \mathbb{D}$ is denoted $\Gamma_{\mathcal{O}}(\mathbb{D}, \mathscr{H}^2)$.

(iii) A section ξ of $\mathscr{H}^{2*} \to \mathbb{D}$ is an assignment

$$\mathbb{D} \ni \tau \mapsto \xi_\tau \in \mathscr{H}^{*2}_\tau.$$

The set of sections is denoted $\Gamma(\mathbb{D}, \mathscr{H}^{2*})$.

(iv) We say that $\xi \in \Gamma(\mathbb{D}, \mathscr{H}^{2*})$ is measurable (resp. smooth, holomorphic) if for each measurable (resp. smooth, holomorphic) section $s \in \Gamma(\mathbb{D}, \mathscr{H}^2)$ the function

$$\mathbb{D} \ni \tau \mapsto \xi_\tau s_\tau \in \mathbb{C}$$

is measurable (resp. smooth, holomorphic). The set of holomorphic sections of $\mathscr{H}^{2*} \to \mathbb{D}$ is denoted $\Gamma_{\mathcal{O}}(\mathbb{D}, \mathscr{H}^{2*})$.

Note that since the bundle $\mathscr{H}^2 \to \mathbb{D}$ is trivial, each $f \in \mathscr{H}^2_o$ defines a holomorphic section

$$\wp^* f \in \Gamma_{\mathcal{O}}(\mathbb{D}, \mathscr{H}^2).$$

It is easy to check that a section $\xi \in \Gamma(\mathbb{D}, \mathscr{H}^{2*})$ is holomorphic if and only if the function

$$\xi(\wp^* f) : \mathbb{D} \ni \tau \mapsto \xi_\tau f$$

is holomorphic.

EXAMPLE 3.2. Let $x \in Y$ and let $v_x \in L_x - \{0\}$. The function

$$\xi^x : \mathscr{H}^2_\tau \ni f \mapsto \frac{f(x)}{v_x} \in \mathbb{C}$$

is linear on each $\mathscr{H}^2_\tau$ and satisfies *Bergman's Inequality*: there exists a C, depending on x, v_x, ω and $e^{-\varphi_\tau}$ such that

$$|\xi^x f|^2 \le C||f||^2_\tau \qquad \text{for all } f \in \mathscr{H}^2_\tau.$$

Thus ξ_x is a section of $\mathscr{H}^{2*} \to \mathbb{D}$ which is evidently holomorphic.

By the Riesz Representation Theorem there exists $k^x_\tau \in \mathscr{H}^2_\tau$ such that

$$\xi^x f = (f, k^x_\tau)_\tau.$$

As is well-known, $K_\tau(x, y) := \overline{k^x_\tau(y)} \otimes v_x$ is the Bergman kernel. ◇

Let $\mathscr{L}^2_\tau$ denote the space of all measurable sections g of $L \to Y$ whose L^2-norm $||g||_\tau$ is finite. For each $\tau \in \mathbb{D}$, $\mathscr{H}^2_\tau$ is a closed subspace of $\mathscr{L}^2_\tau$ and thus there is a bounded orthogonal projection

$$P_\tau : \mathscr{L}^2_\tau \to \mathscr{H}^2_\tau,$$

often called the Bergman projection. In connection with Example 3.2, one can verify that

$$P_\tau f(x) = \int_Y f(y) K_\tau(x, y) e^{-\varphi_\tau(y)} dV_\omega(y).$$

Of course, for any $f \in \mathscr{L}^2_\tau$ the section $f - P_\tau f$ is orthogonal to $\mathscr{H}^2_\tau$ and is therefore the minimal solution of the equation $\bar\partial u = \bar\partial f$. When $\bar\partial f$ is also square integrable one can then apply Hörmander's Theorem to obtain the following estimate for $f - P_\tau f$.

LEMMA 3.3. *Let $f \in \mathscr{L}^2_\tau$ lie in the domain of $\bar\partial$ and let θ be a non-negative $(1,1)$-form such that*

$$\sqrt{-1}\partial_Y \bar\partial_Y \varphi_\tau + \mathrm{Ricci}(\omega) \ge \theta \quad \textit{and} \quad \int_Y |\bar\partial f|^2_\theta e^{-\varphi_\tau} dV_\omega < +\infty.$$

Then

$$\int_Y |f - P_\tau f|^2 e^{-\varphi_\tau} dV_\omega \le \int_Y |\bar\partial f|^2_\theta e^{-\varphi_\tau} dV_\omega.$$

THEOREM 3.4 (Berndtsson's Theorem). *Let Y be a bounded pseudoconvex domain in a Stein Kähler manifold (M, ω), let $L \to M$ be a holomorphic line bundle, and let $e^{-\varphi}$ be a smooth Hermitian metric for the line bundle $\wp^* L \to M \times \mathbb{D}$, where $\wp : M \times \mathbb{D} \to M$ denotes the projection to the first factor. Assume that*

$$\sqrt{-1}(\partial\bar\partial\varphi + \wp^*\mathrm{Ricci}(\omega))$$

is a non-negative $(1,1)$-form on $Y \times \mathbb{D}$. Then for every $\xi \in \Gamma_{\mathcal{O}}(\mathbb{D}, \mathscr{H}^{2*})$ the function

$$\tau \ni \mathbb{D} \mapsto \log ||\xi_\tau||^2_{\tau*}$$

is subharmonic.

Proof. Fix a section $\xi \in \Gamma_{\mathcal{O}}(\mathbb{D}, \mathscr{H}^{2*})$. The goal is to show that the function $\Phi(t) := \log ||\xi_t||^2_{t*}$ is subharmonic, i.e., that

$$\frac{\partial^2}{\partial t \partial \bar t}\Phi(t) \ge 0.$$

Fix $t \in \mathbb{D}$. Then there is a section $f \in \mathscr{H}_t^2$ such that

$$||\xi_t||_{t*}^2 = \frac{|\langle \xi_t, f\rangle|^2}{||f||_t^2}.$$

Define the section $\mathfrak{f} \in \Gamma_{\mathcal{O}}(\mathbb{D}, \mathscr{H}^2)$ by

$$\mathfrak{f}_\tau = f + (\tau - t) P_t \left(\frac{\partial \varphi_t}{\partial t} f \right).$$

This section is holomorphic because it is of the form $\mathfrak{f}_\tau g + \tau h$ for some vectors $f, g \in \mathscr{H}_t^2 = \mathscr{H}_\tau^2$, where the equality of these two vector spaces is their equality as subspaces of $\Gamma_{\mathcal{O}}(Y, L)$. We observe that

$$\mathfrak{f}_t = f \quad \text{and} \quad P_t \left(\frac{\partial \mathfrak{f}_t}{\partial t} - \frac{\partial \varphi_t}{\partial t} \mathfrak{f}_t \right) = -P_t \left(\frac{\partial \varphi_t}{\partial t} f \right) + P_t \left(\frac{\partial \varphi_t}{\partial t} f \right) = 0.$$

Thus the function

$$\Psi(\tau) = \Phi(\tau) - \log \frac{|\langle \xi_\tau, \mathfrak{f}_\tau \rangle|^2}{||\mathfrak{f}_\tau||_\tau^2}$$

is non-negative and vanishes at $\tau = t$. It follows that $\frac{\partial^2}{\partial t \partial \bar{t}} \Psi(t) \ge 0$, which is to say,

$$\frac{\partial^2}{\partial t \partial \bar{t}} \Phi(t) \ge \left. \frac{\partial^2}{\partial \tau \partial \bar{\tau}} \right|_{\tau = t} \log \frac{|\langle \xi_\tau, \mathfrak{f}_\tau \rangle|^2}{||\mathfrak{f}_\tau||_\tau^2}. \tag{6}$$

Thus to prove our result we need to show that the right hand side of (6) is non-negative. Since $\tau \mapsto \langle \xi_\tau, \mathfrak{f}_\tau \rangle$ is holomorphic, it suffices to show that the function $\log \frac{1}{||\xi_\tau||_{\tau*}^2}$ has non-negative laplacian at $\tau = t$.

We compute that

$$\frac{\partial^2}{\partial \tau \partial \bar{\tau}} \log \frac{1}{||\mathfrak{f}_\tau||_\tau^2} = \frac{1}{||\mathfrak{f}_\tau||_\tau^2} \left(- \frac{\partial^2 ||\mathfrak{f}_\tau||_\tau^2}{\partial \tau \partial \bar{\tau}} \right) + \left| \frac{1}{||\mathfrak{f}_\tau||_\tau^2} \frac{\partial ||\mathfrak{f}_\tau||_\tau^2}{\partial \tau} \right|^2.$$

Now,

$$\frac{\partial ||\mathfrak{f}_\tau||_\tau^2}{\partial \tau} = \int_Y \left(\frac{\partial \mathfrak{f}_\tau}{\partial \tau} - \frac{\partial \varphi_\tau}{\partial \tau} \mathfrak{f}_\tau \right) \bar{\mathfrak{f}}_\tau e^{-\varphi_\tau} dV_\omega = \int_Y P_\tau \left(\frac{\partial \mathfrak{f}_\tau}{\partial \tau} - \frac{\partial \varphi_\tau}{\partial \tau} \mathfrak{f}_\tau \right) \bar{\mathfrak{f}}_\tau e^{-\varphi_\tau} dV_\omega$$

and

$$\frac{\partial^2 ||\mathfrak{f}_\tau||_\tau^2}{\partial \tau \partial \bar{\tau}} = - \int_Y \frac{\partial^2 \varphi_\tau}{\partial \tau \partial \bar{\tau}} |\mathfrak{f}_\tau|^2 e^{-\varphi_\tau} dV_\omega + \int_Y \left| \frac{\partial \mathfrak{f}_\tau}{\partial \tau} - \frac{\partial \varphi_\tau}{\partial \tau} \mathfrak{f}_\tau \right|^2 e^{-\varphi_\tau} dV_\omega.$$

By Pythagoras' Theorem and the fact that $P_\tau \frac{\partial \mathfrak{f}_\tau}{\partial \tau} = \frac{\partial \mathfrak{f}_\tau}{\partial \tau}$ we have

$$\begin{aligned}
&\int_Y \left| \frac{\partial \mathfrak{f}_\tau}{\partial \tau} - \frac{\partial \varphi_\tau}{\partial \tau} \mathfrak{f}_\tau \right|^2 e^{-\varphi_\tau} dV_\omega \\
&= \int_Y \left| P_\tau \left(\frac{\partial \mathfrak{f}_\tau}{\partial \tau} - \frac{\partial \varphi_\tau}{\partial \tau} \mathfrak{f}_\tau \right) \right|^2 e^{-\varphi_\tau} dV_\omega + \int_Y \left| \left(\frac{\partial \mathfrak{f}_\tau}{\partial \tau} - \frac{\partial \varphi_\tau}{\partial \tau} \mathfrak{f}_\tau \right) - P_\tau \left(\frac{\partial \mathfrak{f}_\tau}{\partial \tau} - \frac{\partial \varphi_\tau}{\partial \tau} \mathfrak{f}_\tau \right) \right|^2 e^{-\varphi_\tau} dV_\omega \\
&= \int_Y \left| P_\tau \left(\frac{\partial \mathfrak{f}_\tau}{\partial \tau} - \frac{\partial \varphi_\tau}{\partial \tau} \mathfrak{f}_\tau \right) \right|^2 e^{-\varphi_\tau} dV_\omega + \int_Y \left| \frac{\partial \varphi_\tau}{\partial \tau} \mathfrak{f}_\tau - P_\tau \left(\frac{\partial \varphi_\tau}{\partial \tau} \mathfrak{f}_\tau \right) \right|^2 e^{-\varphi_\tau} dV_\omega.
\end{aligned}$$

We therefore have

$$\begin{aligned}
&\frac{\partial^2}{\partial\tau\partial\bar\tau}\log\frac{1}{||\mathfrak{f}_\tau||^2_\tau}\\
&=\frac{1}{||\mathfrak{f}_\tau||^2_\tau}\left(\int_Y\frac{\partial^2\varphi_\tau}{\partial\tau\partial\bar\tau}|\mathfrak{f}_\tau|^2e^{-\varphi_\tau}dV_\omega-\left|\left|\frac{\partial\varphi_\tau}{\partial\tau}\mathfrak{f}_\tau-P_\tau\left(\frac{\partial\varphi_\tau}{\partial\tau}\mathfrak{f}_\tau\right)\right|\right|^2_\tau\right)\\
&\qquad+\frac{\left|\left(P_\tau\left(\frac{\partial\mathfrak{f}_\tau}{\partial\tau}-\frac{\partial\varphi_\tau}{\partial\tau}\mathfrak{f}_\tau\right),\mathfrak{f}_\tau\right)_\tau\right|^2}{||\mathfrak{f}_\tau||^4_\tau}-\frac{\left|\left|P_\tau\left(\frac{\partial\mathfrak{f}_\tau}{\partial\tau}-\frac{\partial\varphi_\tau}{\partial\tau}\mathfrak{f}_\tau\right)\right|\right|^2_\tau}{||\mathfrak{f}_\tau||^2_\tau},
\end{aligned}$$

and setting $\tau=t$ we obtain

$$\left.\frac{\partial^2}{\partial\tau\partial\bar\tau}\right|_{\tau=t}\log\frac{|\langle\xi_\tau,\mathfrak{f}_\tau\rangle|^2}{||\mathfrak{f}_\tau||^2_\tau}=\frac{1}{||\xi_t||^2_{t*}}\left(\int_Y\frac{\partial^2\varphi_t}{\partial t\partial\bar t}|f|^2e^{-\varphi_t}dV_\omega-\left|\left|\frac{\partial\varphi_t}{\partial t}f-P_t\left(\frac{\partial\varphi_t}{\partial t}f\right)\right|\right|^2_t\right)$$

By Lemma 3.3 we have the estimate

$$\int_Y\left|\frac{\partial\varphi_t}{\partial t}f-P_t\left(\frac{\partial\varphi_t}{\partial t}f\right)\right|^2e^{-\varphi_t}dV_\omega\le\int_Y\left|\bar\partial_Y\frac{\partial\varphi_t}{\partial t}\right|^2_{\theta_t}|f|^2e^{-\varphi_t}dV_\omega$$

where we have set $\theta_t:=\sqrt{-1}\partial_Y\bar\partial_Y\varphi_t+\mathrm{Ricci}(\omega)$. We therefore obtain the estimate

$$\left.\frac{\partial^2}{\partial\tau\partial\bar\tau}\log\frac{1}{||\xi_\tau||^2_{\tau*}}\right|_{\tau=t}\ge\frac{1}{||\xi_t||^2_{t*}}\int_Y\left(\frac{\partial^2\varphi_t}{\partial t\partial\bar t}-\left|\bar\partial_Y\frac{\partial\varphi_t}{\partial t}\right|^2_{\theta_t}\right)|f|^2e^{-\varphi_t}dV_\omega.$$

Finally, if A is a Hermitian $n\times n$ matrix, $v\in\mathbb{C}^n$ and $c\in\mathbb{R}$ then one has the determinant formula

$$\det\begin{pmatrix}c & v\\ v^\dagger & A\end{pmatrix}=\left(c-(A^{-1}v,v)\right)\det A,$$

which reads as

$$\frac{\left(\sqrt{-1}\partial\bar\partial\varphi+\wp^*\mathrm{Ricci}(\omega)\right)^{n+1}}{(n+1)!}=\left(\frac{\partial^2\varphi_t}{\partial t\partial\bar t}-\left|\bar\partial_Y\frac{\partial\varphi_t}{\partial t}\right|^2_{\theta_t}\right)\frac{\theta_t^n}{n!}\wedge\sqrt{-1}dt\wedge d\bar t$$

in our setting. Thus the positivity of $\sqrt{-1}\partial\bar\partial\varphi+\wp^*\mathrm{Ricci}(\omega)$ implies that

$$\frac{\partial^2\varphi_t}{\partial t\partial\bar t}-\left|\bar\partial_Y\frac{\partial\varphi_t}{\partial t}\right|^2_{\theta_t}\ge 0,$$

and the proof is complete. □

3.2. **Berndtsson's Theorem in the Disk Bundle.** This section explicates a version of Berndtsson's Theorem 3.4 adapted to the disk bundle $\mathscr{B}(\lambda)$. This version of Berndtsson's Theorem will be used in Section 5, in the conclusion of the proof of Theorem 1.

As already mentioned in Paragraph 2.1, the metric $e^{-\lambda}$ is non-negatively curved if and only if the disk bundle $\mathscr{B}(\lambda)$ is pseudoconvex. It is not difficult to show that in fact, $\mathscr{B}(\lambda)$ is a pseudoconvex domain in the larger Stein manifold $\tilde Y:=\{v\in L^*\ ;\ \pi v\in Y \text{ and } |v|^2e^\lambda<1+\varepsilon\}$. Indeed, one can take a smooth, strictly plurisubharmonic exhaustion function Ψ for Y and consider the exhaustion function

$$\tilde\Psi:=\pi^*\Psi-\log(1+\varepsilon-|v|^2e^\lambda)$$

for $\tilde{Y}$. This function is strictly plursubharmonic provided Ψ has sufficiently large Hessian— a condition that can be arrange by composing Ψ with a sufficiently rapidly increasing convex function.

The domain $\mathscr{B}(\lambda)$ admits the singular Hermitian metric

$$\frac{e^{-\tilde{\varphi}+m\tilde{\lambda}}}{(|\sigma|^2 e^{\tilde{\lambda}})^{1-k-\delta_o}}$$

for the line bundle

$$\Lambda_m := \pi^*(L \otimes L_Z^{*\otimes m}).$$

The curvature of this metric is

$$\partial\bar{\partial}\varphi - (k+m-1+\delta_o)\partial\bar{\partial}\lambda + [\mathbb{O}_{L_Z^*}],$$

where the last term is the current of integration over the zero section $\mathbb{O}_{L_Z^*}$ of L_{Z^*}, seen as a divisor in $\mathscr{B}(\lambda)$. There is also the smooth measure $d\mathscr{V}$ associated to the to the $(1,1)$-form

$$\tilde{\omega} := \frac{1}{c_o^{1/(n+1)}}\left(\pi^*\omega + c_o\sqrt{-1}\partial\bar{\partial}|v|^2 e^{\lambda}\right),$$

which is a Kähler form provided $c_o > 0$ is sufficiently small. Observe that

$$d\mathscr{V} := \frac{\tilde{\omega}^{n+1}}{(n+1)!} := \frac{1}{c_o}\left(\pi^*\omega + c_o\sqrt{-1}\partial\bar{\partial}|v|^2 e^{\lambda}\right)^{n+1} = \frac{\pi^*\omega^n}{n!} \wedge dd^c|v|^2 e^{\lambda},$$

and hence the Ricci curvature of $\tilde{\omega}$ is

$$-\partial\bar{\partial}\log d\mathscr{V} = \pi^*(\mathrm{Ricci}(\omega) - \partial\bar{\partial}\lambda).$$

Finally, let us also fix a function $U \in \mathrm{PSH}(\mathscr{B}(\lambda) \times \mathbb{D})$. We write

$$U_\tau(v) := U(v,\tau).$$

To simplify large formulas, we sometimes write

$$\psi_\tau := \tilde{\varphi} + U_\tau - m\tilde{\lambda} + (1-k-\delta_o)\log|\sigma|^2 e^{\tilde{\lambda}}.$$

Define the trivial vector bundle $\mathscr{H}^2 \to \mathbb{D}$ whose fibers are

$$\mathscr{H}^2_\tau := \left\{F \in \Gamma_{\mathcal{O}}(\mathscr{B}(\lambda), \Lambda_m)\ ;\ \int_{\mathscr{B}(\lambda)} |F|^2 e^{-\psi_\tau} d\mathscr{V} < +\infty\right\},$$

equipped with the L^2 metric on the fibers. Since

$$\partial\bar{\partial}\psi + \mathrm{Ricci}(\tilde{\omega}) \geq \pi^*(\partial\bar{\partial}\varphi + \mathrm{Ricci}(\omega) - (k+1+m-1+\delta_o)\partial\bar{\partial}\lambda)$$

the following result holds.

THEOREM 3.5 (Berndtsson's Theorem. Disk Bundle Case). *Let X be a bounded pseudoconvex domain in a Stein Kähler manifold (Y,ω), let $L \to Y$ be a holomorphic line bundle with smooth Hermitian metric $e^{-\varphi}$. Let $Z \subset Y$ be a smooth hypersurface and assume there exist a section $T \in \Gamma_{\mathcal{O}}(Y, L_Z)$ and a smooth metric $e^{-\lambda}$ for $L_Z \to Y$ such that*

$$\sup_Y |T|^2 e^{-\lambda} \leq 1 \quad \textit{and} \quad dd^c\lambda \geq 0.$$

Suppose, moreover, that

$$\sqrt{-1}(\partial\bar{\partial}\varphi - (k+1+m-1+\delta_o)\partial\bar{\partial}\lambda + \mathrm{Ricci}(\omega))$$

is a non-negative $(1,1)$*-form on* X*. Let* $\mathscr{H}^2 \to \mathbb{D}$ *be the vector bundle just defined. Then for every holomorphic section* $\xi \in \Gamma_{\mathcal{O}}(\mathbb{H}, \mathscr{H}^{2*})$ *of the dual bundle* $\mathscr{H}^{2*} \to \mathbb{H}$ *the function*

$$\tau \mapsto \log ||\xi_\tau||^2_{\tau *}$$

is subharmonic on $\mathbb{D}$*.*

4. Extension of normal jets from a flat hypersurface

This section establishes the following slight generalization of a special case of Theorem 1.

THEOREM 4.1. *Let* (X, ω) *be a Stein Kähler manifold and* $Z \subset X$ *a possibly singular complex hypersurface cut out by a holomorphic function* $T \in \mathcal{O}(X)$ *satisfying*

$$\sup_X |T|^2 < 1$$

for some smooth function $\lambda : X \to \mathbb{R}$*. Let* $L \to X$ *be a holomorphic line bundle with singular Hermitian metric* $e^{-\varphi}$ *such that*

$$\partial\bar{\partial}\varphi + \mathrm{Ricci}(\omega) \geq 0.$$

Then for every $(k+1)$*-tuple of sections* $(f_0, ..., f_k) \in (\Gamma_{\mathcal{O}}(Z, L))^{k+1}$ *satisfying*

$$\sum_{j=0}^{k} \int_Z \frac{|f_j|^2 e^{-\varphi}}{|dT|^2_\omega} dA_\omega < +\infty$$

there exists $F \in \Gamma_{\mathcal{O}}(X, L)$ *such that*

$$F^{(j)}_{\perp} = f_j, \ \ 0 \leq j \leq k, \quad \textit{and} \quad \int_X |F|^2 e^{-\varphi} dV_\omega \leq C^k \sum_{j=0}^{k} \int_Z \frac{|f_j|^2 e^{-\varphi}}{|dT|^2_\omega} dA_\omega,$$

where C *is a universal constant.*

REMARK 4.2. Part of the proof of Theorem 1 applies Theorem 4.1 to the singular hypersurface

$$\mathcal{Z} := \pi^{-1}(Z) \cup \mathbb{O}_{L^*_Z} \subset \mathscr{B}(\lambda),$$

where $\mathbb{O}_{L^*_Z}$ denotes the image of zero section of $L^*_Z \to X$ in the total space of L^*_Z. Note that $\mathbb{O}_{L^*_Z}$ is precisely the zero locus of the diagonal section $\sigma \in \Gamma_{\mathcal{O}}(L^*_Z, \pi^* L^*_Z)$ defined in Proposition 2.1 (for the line bundle $H = L_Z$). Thus the holomorphic function

$$\mathscr{T} := \langle \pi^* T, \sigma \rangle \,, \tag{7}$$

defined by the canonical pairing of the dual line bundles $\pi^* L_Z$ and $\pi^* L^*_Z$, cuts out $\mathcal{Z}$. Observe that

$$\sup_{\mathscr{B}(\lambda)} |\mathscr{T}|^2 \leq \sup_{\mathscr{B}(\lambda)} \pi^* |T|^2 e^{-\lambda} |\sigma|^2 e^{\lambda} \leq 1.$$

The fact that the hypersurface $\mathcal{Z}$ is singular accounts for the need to establish Theorem 4.1 for singular hypersurfaces. ◇

It suffices to assume that X is a smoothly bounded pseudoconvex domain in a larger Stein manifold, that the smooth locus of Z meets ∂X transversely, that the line bundle L is defined on a neighborhood of X, and that the metric $e^{-\varphi}$ is smooth up to the boundary of X. We may also assume that the normal k-jet $(f_0, ..., f_k) \in (\Gamma_{\mathcal{O}}(Z, L))^{k+1}$ extends holomorphically to a neighborhood of $\overline{Z}$. Indeed, if Theorem 4.1 is proved for such data, then standard compactness and limit theorems from real analysis yield the result in the general case.

4.1. **Dual formulation of the norm of the minimal extension.** Write

$$\mathscr{H}^2(\omega,\varphi) := \left\{ F \in \Gamma_{\mathcal{O}}(X,L)\ ;\ \int_X |F|^2 e^{-\varphi} dV_\omega < +\infty \right\}$$

and define

$$\mathfrak{J}^k_Z(\omega,\varphi) := \left\{ G \in \mathscr{H}^2(\omega,\varphi)\ ;\ G^{(j)}_\perp = 0 \text{ for } 0 \le j \le k \right\}$$

and

$$\mathrm{Ann}(\mathfrak{J}^k_Z(\omega,\varphi)) := \left\{ \xi \in \mathscr{H}^2(\omega,\varphi)\ ;\ \langle \xi, G\rangle = 0 \text{ for all } G \in \mathfrak{J}^k_Z(\omega,\varphi) \right\}.$$

By Proposition 2.5 there exists $G \in \Gamma_{\mathcal{O}}(\overline{X}, L)$

$$G^{(j)}_\perp = f_j, \quad 0 \le j \le k.$$

Since G is holomorphic on a neighborhood of X,

$$\int_X |G|^2 e^{-\varphi} dV_\omega < +\infty.$$

Let F_o denote the element of $\mathscr{H}^2(\omega,\varphi)$ of minimal norm whose normal k-jet along Z is $(f_o, ..., f_k)$. Since any two normal k-jet extensions of $(f_o, ..., f_k)$ differ by an element of $\mathfrak{J}^k_Z(\omega,\varphi)$, the extension of minimal norm is orthogonal to $\mathfrak{J}^k_Z(\omega,\varphi)$. Thus, following Berndtsson and Lempert, one has the following dual formulation of the norm of the minimal normal k-jet extension.

PROPOSITION 4.3. *If $(f_o, ..., f_k) \in (\Gamma_{\mathcal{O}}(Z,L))^{k+1}$ is the normal k-jet along Z of some section $F \in \mathscr{H}^2(\omega,\varphi)$ then the minimal extension $F_o \in \mathscr{H}^2(\omega,\varphi)$ of $(f_o, ..., f_k)$ satisfies*

$$\int_X |F_o|^2 e^{-\varphi} dV_\omega = \sup \left\{ \frac{|\langle \xi, F\rangle|^2}{||\xi||^2_*}\ ;\ \xi \in \mathrm{Ann}(\mathfrak{J}^k_Z(\omega,\varphi)) \right\}. \tag{8}$$

Note that the right hand side of (8) is independent of the choice of F, as it must be. To estimate the right hand side of (8), we use the following lemma.

LEMMA 4.4. *The set of linear functionals in $\mathscr{H}^{2*}_o$ of the form*

$$\xi_g : \mathscr{H}^2(\omega,\varphi) \ni F \mapsto \sum_{j=0}^k \int_Z F^{(j)}_\perp \overline{g_j} e^{-\varphi} dA_\omega, \quad g = (g_o, ..., g_k) \in (\Gamma_o(Z,L))^{k+1},$$

forms a dense subspace of $\mathrm{Ann}(\mathfrak{J}_\varphi(Z))$.

(Here $\Gamma_o(Z,L)$ means smooth sections of L with compact support on the smooth locus of Z.)

Proof. Clearly $\xi_g \in \mathrm{Ann}(\mathfrak{J}_Z(\omega,\varphi))$. Now suppose $\xi \in \mathscr{H}^2(\omega,\varphi)^*$ is not in the closure of the subspace $V = \{\xi_g\ ;\ g \in (\Gamma_o(Z,L))^{k+1}\}$. Then there is a bounded linear functional

$$F \in \mathscr{H}^2(\omega,\varphi)^{**} = \mathscr{H}^2(\omega,\varphi)$$

such that

$$\langle \xi, F\rangle \ne 0 \quad \text{and} \quad \langle \xi_g, F\rangle = 0 \text{ for all } g \in (\Gamma_o(Z,L))^{k+1}.$$

The latter condition implies that $F|_Z \equiv 0$, and thus $\xi \notin \mathrm{Ann}(\mathfrak{J}_Z(\omega,\varphi))$, as required. □

COROLLARY 4.5. *If* $(f_o, ..., f_k) \in (\Gamma_{\mathcal{O}}(Z,L))^{k+1}$ *is the normal* k*-jet along* Z *of some section* $F \in \mathscr{H}^2(\omega,\varphi)$ *then the minimal extension* $F_o \in \mathscr{H}^2(\omega,\varphi)$ *of* $(f_o, ..., f_k)$ *satisfies*

$$\int_X |F_o|^2 e^{-\varphi} dV_\omega = \sup\left\{ \frac{|\langle \xi_g, F\rangle|^2}{||\xi_g||^2_*} \; ; \; g \in (\Gamma_o(Z,L))^{k+1} \right\}. \tag{9}$$

4.2. **Extension of a normal k-jet to a small neighborhood of Z.** Let

$$X_t := \{x \in X \; ; \; \log|T(x)|^2 < t\}, \quad t \le 0.$$

Then $X_o = X$ and for $t << 0$ X_t is a tubular neighborhood of Z. (Recall that X is bounded and all the data is smooth up to the boundary.) The first step in our proof of Theorem 4.1 is to establish the following theorem.

THEOREM 4.6. *Let* $L \to X_t$ *be a holomorphic line bundle with smooth Hermitian metric* $e^{-\varphi}$ *have non-negative curvature, and assume all the data extends smoothly up to the boundary of* X_t*. Let* $(f_o, ..., f_k) \in (\Gamma_{\mathcal{O}}(Z,L))^{k+1}$ *satisfy*

$$\sum_{j=0}^k \int_Z \frac{|f_j|^2 e^{-\varphi}}{|dT|^2_\omega} dA_\omega < +\infty.$$

Then for $t << 0$ *there exists* $F_t \in \Gamma_{\mathcal{O}}(X_t, L)$ *such that*

$$F^{(j)}_\perp = f_j, \; 0 \le j \le k \quad \textit{and} \quad e^{-t}\int_{X_t} |F|^2 e^{-\varphi} dV_\omega \le C^k \sum_{j=0}^k \int_Z \frac{|f_j|^2 e^{-\varphi}}{|dT|^2_\omega} dA_\omega$$

for some universal constant $C > 0$.

One begins with the well-known Twisted Basic Estimate. A proof can be found in [MV-2007].

THEOREM 4.7. *Let* (X,ω) *be a complete Kähler manifold and* $L \to X$ *a holomorphic line bundle with smooth Hermitian metric* $e^{-\kappa}$*. Fix positive functions* $\tau \in \mathscr{C}^2(X)$ *and* $A : X \to (0,\infty)$*. Then for every* L*-valued* $(0,1)$*-form* $\beta \in \mathrm{Domain}(\bar\partial) \cap \mathrm{Domain}(\bar\partial^*)$ *we have*

$$\int_X (\tau + A)|\bar\partial^*\beta|^2 e^{-\kappa} + \int_X \tau|\bar\partial\beta|^2_\omega e^{-\varphi}$$

$$\ge \int_X \left\langle \left(\tau(\partial\bar\partial\kappa + \mathrm{Ricci}(\omega)) - \partial\bar\partial\tau - A^{-1}\partial\tau \wedge \bar\partial\tau\right), V_\beta \wedge \overline{V_\beta}\right\rangle e^{-\kappa}, \tag{10}$$

where

$$V_\beta := g^{i\bar j}\beta_{\bar j} \otimes \frac{\partial}{\partial z^i}$$

is the L*-valued* $(1,0)$*-vector field induced by* β *and the Kähler metric.*

Following [MV-2007], Theorem 4.7 is used to obtain an *a priori* estimate by making a good choice of τ, A and κ, as follows. First let

$$\tau = a + h(a) \quad \text{and} \quad A = \frac{(1 + h'(a))^2}{-h''(a)},$$

where

$$h(x) := 2 - x + \log(2e^{x-1} - 1) \quad \text{and} \quad a = \gamma - \log(|T|^2 + e^t),$$

with $\gamma = 1 + \log 2 + t$, so that $a \geq 1$. Thus

$$\tau \geq 1, \quad 1 + h'(a) = \frac{2e^{a-1}}{2e^{a-1} - 1} \geq 1 \quad \text{and} \quad A = 2e^{a-1}.$$

Then

$$-\partial\bar{\partial}a = \frac{e^t dT \wedge \overline{dT}}{(|T|^2 + e^t)^2}$$

and thus

$$-\partial\bar{\partial}\tau - \frac{\partial\tau \wedge \bar{\partial}\tau}{A} = (1 + h'(a))(-\partial\bar{\partial}a) \geq \frac{e^t dT \wedge \overline{dT}}{(|T|^2 + e^t)^2}.$$

Finally take

$$\kappa = \varphi + (k+1)\log|T|^2 - k\log\left|T - 2e^{t/2}\right|^2,$$

which satisfies

$$\tau(\partial\bar{\partial}\kappa + \mathrm{Ricci}(\omega)) \geq 0$$

because $|T|^2 < e^t$ in X_t.

REMARK 4.8. This choice of the metric $e^{-\kappa}$ is absolutely crucial to our proof of the jet extension theorem. Indeed, note that any function that is locally integrable with respect to the weight $e^{-\kappa}$ must vanish to order $k+1$ along Z. The last term of κ, which is the negative of the logarithm of the norm of a holomorphic function, is introduced In order get control on the size of $e^{-\kappa}$. In the manifold X κ is not positively curved, but the curvature is indeed positive in the slab X_t, and this is precisely the reason for beginning with extension to a thin neighborhood of Z. ◇

With the choices of τ, A and κ just made, Theorem 4.7 implies the following result.

THEOREM 4.9. *Let $\mathfrak{D} := \bar{\partial} \circ \sqrt{\tau + A}$ and $S = \sqrt{\tau} \circ \bar{\partial}$. Then*

$$\int_{X_t} |\mathfrak{D}^*\beta|^2 e^{-\kappa} dV_\omega + \int_{X_t} |S\beta|^2_\omega e^{-\kappa} dV_\omega \geq \int_{X_t} \frac{e^t\,|dT(V_\beta)|^2}{(|T|^2 + e^t)^2} e^{-\kappa} dV_\omega$$

for any L-valued $(0,1)$-form β in the domain of $\mathscr{D}^*$ and S.

As a corollary, the usual L^2 method establishes the following theorem.

THEOREM 4.10. *Let α by a measurable, L-valued $(0,1)$-form on X_t such that $S\alpha = 0$ in the sense of currents, and assume there exists a constant $C > 0$ such that*

$$\left|\int_{X_t} \langle\alpha, \beta\rangle_\omega e^{-\kappa} dV_\omega\right| \leq C\left(\int_{X_t} |\mathfrak{D}^*\beta|^2 e^{-\kappa} dV_\omega + \int_{X_t} |S\beta|^2_\omega e^{-\kappa} dV_\omega\right)$$

for all β in the domains of $\mathfrak{D}^*$ and S. Then there exists a section u of $L \to X_t$ such that

$$\mathfrak{D}u = \alpha \quad \textit{and} \quad \int_{X_t} |u|^2 e^{-\kappa} dV_\omega \leq C.$$

Moreover, if α is smooth then so is u, and thus u vanishes to order $k+1$ on Z.

Theorem 4.10 is applied to the form α constructed as follows. As in the proof of Proposition 2.5, fix holomorphic sections $g_0, ..., g_k \in \Gamma_{\mathscr{O}}(X_t, L)$ such that

$$g_j = f_j + O(|T|^{k+1}) \text{ near } Z, \quad 0 \leq j \leq k.$$

Set

$$G := \sum_{j=0}^{k} g_j T^j.$$

Let $\chi \in \mathscr{C}_o^\infty([0,1))$ be a function such that $\chi \equiv 1$ on $[0,1/2]$ and $|\chi'| \le 2$. Write

$$\chi_t = \chi((|T|^2/e^t)$$

and set

$$\alpha_t := G\bar{\partial}\chi_t = G \cdot \chi'\left(\tfrac{|T|^2}{e^t}\right)\frac{T\overline{dT}}{e^t}.$$

Then

$$\begin{aligned}
|(\alpha_t, \beta)|^2 &\le \left(\int_{X_t} |\langle \alpha_t, \beta\rangle_\omega| \, e^{-\kappa} dV_\omega\right)^2 \\
&= \left(\int_{X_t} \left| G\chi'\left(\tfrac{|T|^2}{e^t}\right)\frac{T}{e^t}\overline{dT(V_\beta)}_\omega\right| e^{-\kappa} dV_\omega\right)^2 \\
&\le \left(\int_{X_t} \left|G\chi'(\tfrac{|T|^2}{e^t})\right|^2 \frac{(|T|^2+e^t)^2}{e^{3t}} |T|^2 e^{-\kappa} dV_\omega\right)\left(\int_{X_t} \frac{e^t\,|dT(V_\beta)|^2}{(|T|^2+e^t)^2} e^{-\kappa} dV_\omega\right).
\end{aligned}$$

The first factor in the last term satisfies

$$\begin{aligned}
&\int_{X_t} \left|G\chi'(\tfrac{|T|^2}{e^t})\right|^2 \frac{(|T|^2+e^t)^2}{e^{3t}} e^{-\kappa} dV_\omega \\
&\le \frac{16}{e^t}\int_{\frac{1}{2}e^t \le |T|^2 \le e^t} \frac{|T-2e^{t/2}|^{2k}}{|T|^{2k}} \left|\sum_{j=0}^{k} g_j T^j\right|^2 e^{-\varphi} dV_\omega \\
&\le \frac{16}{e^t} k \int_{\frac{1}{2}e^t \le |T|^2 \le e^t} \left(\sum_{j=0}^{k} |g_j|^2 \frac{|T-2e^{t/2}|^{2k}}{|T|^{2(k-j)}}\right) e^{-\varphi} dV_\omega \\
&\le 16\pi k 6^k \frac{1}{\pi e^t}\int_{\frac{1}{2}e^t \le |T|^2 \le e^t} \left(\sum_{j=0}^{k} |g_j|^2 e^{-\varphi}\right) dV_\omega \\
&\le 20\pi k 6^k \left(\sum_{j=0}^{k} \int_Z \frac{|f_j|^2 e^{-\varphi}}{|dT|^2_\omega} dA_\omega\right) \\
&\le (120\pi)^k \left(\sum_{j=0}^{k} \int_Z \frac{|f_j|^2 e^{-\varphi}}{|dT|^2_\omega} dA_\omega\right),
\end{aligned}$$

provided t is sufficiently small. (The second to last estimate holds even for singular varieties, provided a very small constant is added, because the singular locus has measure zero. At the end of the process, one can take the limit as this constant goes to 0; this procedure is rather standard, so details are left to the reader.) By Theorem 4.10 there exists a smooth section u of $L \to X_t$ satisfying the estimate

$$\int_{X_t} |u|^2 e^{-\kappa} dV_\omega \le (120\pi)^k \left(\sum_{j=0}^{k} \int_Z \frac{|f_j|^2 e^{-\varphi}}{|dT|^2_\omega} dA_\omega\right),$$

and in particular u vanishes to order $k+1$ along Z. Define

$$F_t := G\chi_t - \sqrt{\tau + A}u \in \Gamma_{\mathscr{O}}(X_t, L).$$

Then

$$(F_t)^{(j)}_{\perp} = f_j, \quad 0 \le j \le k$$

and

$$\int_{X_t} |F_t|^2 e^{-\varphi} dV_\omega \le 2\int_{X_t} |G|^2 e^{-\varphi} dV_\omega + 2\int_{X_t} (\tau + A)|T|^2 \frac{|T|^{2k}}{|T - 2e^{t/2}|^{2k}} |u|^2 e^{-\kappa} dV_\omega$$

Now, if $t << 0$ then

$$\int_{X_t} |G|^2 e^{-\varphi} dV_\omega \le (k+1)\sum_{j=0} \int_{X_t} |g_j|^2 |T|^{2j} e^{-\varphi} dV_\omega \le C_o(k+1)e^t \sum_{j=0} \int_Z \frac{|f_j|^2 e^{-\varphi}}{|dT|^2} dA_\omega.$$

Turning to the estimate of the second term, observe that on X_t

$$\frac{|T|^{2k}}{|T - 2e^{t/2}|^{2k}} \le 1,$$

and since $a \ge 1$,

$$(\tau + A)|T|^2 \le (\tau + A)e^{\gamma - a} = (2 - \log(2e^{a-1} - 1) + 2e^{a-1})e^{\gamma - a} \le e^{\gamma - 1}(2e^{1-a} + 4) \le 6e^{\gamma - 1}.$$

Thus

$$\int_{X_t} (\tau + A)|T|^2 \frac{|T|^{2k}}{|T - 2e^{t/2}|^{2k}} |u|^2 e^{-\kappa} dV_\omega \le 6e^{\gamma - 1} \int_{X_t} |u|^2 e^{-\kappa} dV_\omega = 12e^t \int_{X_t} |u|^2 e^{-\kappa} dV_\omega.$$

Thus there is a universal constant C such that

$$e^{-t} \int_{X_t} |F_t|^2 e^{-\varphi} dV_\omega \le C^k \left(\sum_{j=0}^{k} \int_Z \frac{|f_j|^2 e^{-\varphi}}{|dT|^2_\omega} dA_\omega \right).$$

The proof of Theorem 4.6 is therefore complete. □

4.3. **Degeneration to the infinitesimal neighborhood.** Following Berndtsson and Lempert, let

$$\mathbb{H} := \{\tau \in \mathbb{C} \, ; \, \mathrm{Re}\, \tau < 0\}$$

denote the left half plane, and define the plurisubharmonic function $U : X \times \mathbb{H} \to [-\infty, 0)$ by

$$U(x, \tau) := \max(\log |T(x)|^2 - \mathrm{Re}\, \tau, 0).$$

We may sometimes write

$$U_\tau(x) := U(x, \tau).$$

Define a family of metrics $\{e^{-\psi_{\tau,p}} \, ; \, \tau \in \mathbb{H}, \, p > 0\}$ for $L \to X$ by

$$e^{-\psi_{\tau,p}} = e^{-(\varphi + pU_\tau)},$$

and define Hilbert spaces

$$\mathscr{H}^2_{\tau,p} := \left\{ F \in \Gamma_{\mathscr{O}}(X, L) \, ; \, ||F||^2_{\tau,p} := e^{-\mathrm{Re}\, \tau} \int_X |F|^2 e^{-\psi_{\tau,p}} dV_\omega < +\infty \right\}.$$

Observe that

$$p < p' \Rightarrow ||F||^2_{\tau,p'} \le ||F||^2_{\tau,p} \Rightarrow \mathscr{H}^2_{\tau,p} \subset \mathscr{H}^2_{\tau,p'},$$

and that, since

$$e^{-\mathrm{Re}\ \tau}\int_{X_{\mathrm{Re}\ \tau}}|F|^2e^{-\varphi}dV_\omega \le ||F||^2_{\tau,p} \quad \text{for all } \tau\in\mathbb{H},$$

$\mathscr{H}^2_{\tau,p}\subset\mathbf{V}^2_\tau$, where

$$\mathbf{V}^2_\tau = \left\{F\in\Gamma_{\mathcal{O}}(X,L)\ ;\ e^{-\mathrm{Re}\ \tau}\int_{X_{\mathrm{Re}\ \tau}}|F|^2e^{-\varphi}dV_\omega < +\infty\right\}.$$

On the other hand given, $F\in\mathscr{H}^2_{\tau,p}$,

$$\lim_{p\to\infty}||F||^2_{\tau,p} = e^{-\mathrm{Re}\ \tau}\int_{X_{\mathrm{Re}\ \tau}}|F|^2e^{-\varphi}dV_\omega.$$

Consequently, Theorem 4.6 implies that the extension of minimal norm $F_t\in\mathscr{H}^2_{t+\sqrt{-1}s,p}$ satisfies

$$||F_t||^2_{t,p}\le C^k\sum_{j=o}^{k}\int_Z\frac{|f_j|^2e^{-\varphi}}{|dT|^2_\omega}dA_\omega.$$

Defining

$$\mathfrak{I}^k_{\tau,p}(Z) := \left\{G\in\mathscr{H}^2_{\tau,p}\ ;\ G^{(j)}_\perp = 0 \text{ for } 0\le j\le k\right\}$$

and

$$\mathrm{Ann}(\mathfrak{I}^k_{\tau,p}(Z)) := \left\{\xi\in\mathscr{H}^2(\omega,\varphi)\ ;\ \langle\xi,G\rangle = 0 \text{ for all } G\in\mathfrak{I}^k_Z(\omega,\varphi)\right\},$$

one has, as in Corollary 4.5,

$$e^{-t}\int_X|F_t|^2e^{-\psi_{t,p}}dV_\omega = \sup\left\{\frac{|\langle\xi_g,F\rangle|^2}{||\xi_g||^2_{t,p*}}\ ;\ g\in(\Gamma_o(Z,L))^{k+1}\right\}. \tag{11}$$

In the expression

$$\frac{|\langle\xi_g,F\rangle|^2}{||\xi_g||^2_{t,p*}}$$

the numerator is independent of t. By Berndtsson's Theorem, the function

$$(-\infty,0)\ni t\mapsto\log||\xi_g||^2_{t,p*}$$

is convex.

LEMMA 4.11. *For each $g\in(\Gamma_o(Z,L))^{k+1}$ there exists a constant C_g such that*

$$\limsup_{t\to-\infty}||\xi_g||^2_{t,p*}\le C_g.$$

Proof. Let $H\in\mathscr{H}^2_{t,p}$ satisfy

$$e^{-t}\int_X|H|^2e^{-\psi_{t,p}}dV_\omega = 1.$$

Associate to H the normal k-jet $(h_o,...,h_k)\in\Gamma_{\mathcal{O}}(Z,L)$ where

$$h_j(z) = \frac{1}{2\pi\sqrt{-1}}\int_{|T|=\frac{1}{2}e^{t/2}}H(z,T)\frac{dT}{T^{j+1}}.$$

Here T is used as a local coordinate near Z, and z as coordinates along Z. This formula shows that h_j are controlled by the values of H in a small neighborhood of Z. Thus using Bergman's Inequality one obtains the estimate

$$|h_j(z)|^2 e^{-\varphi(z,0)} \le \tilde{C}_g e^{-t} \int_{X_t} |H|^2 e^{-\varphi} dV_\omega \le \tilde{C}_g,$$

and consequently by Cauchy-Schwarz,

$$||\xi_g||^2_{t,p*} = \sup_{||H||_{t,p}=1} \left| \sum_{j=0}^{k} \int_{\text{Support}(g_j)} \frac{h_j \bar{g}_j e^{-\varphi}}{|dT|^2_\omega} dA_\omega \right|^2 \le C_g,$$

as claimed. □

In view of Lemma 4.11 the function $t \mapsto \log ||\xi_g||^2_{t,p*}$ is bounded, and since it is also convex, it must be increasing. It follows that for $t < 0$,

$$\int_X |F_o|^2 e^{-\varphi} dV_\omega \le e^{-t} \int_X |F_t|^2 e^{-\psi_{t,p}} dV_\omega.$$

In particular, for $t << 0$ small enough, Theorem 4.6 holds, and thus one has the estimate

$$\int_X |F_o|^2 e^{-\varphi} dV_\omega \le C^k \sum_{j=o}^{k} \frac{|f_j|^2 e^{-\varphi}}{|dT|^2_\omega} dA_\omega.$$

The proof of Theorem 4.1 is therefore complete. □

5. End of the proof of Theorem 1

5.1. **Lifting to the disk bundle.** As already suggested in Remark 4.2, the general case of Theorem 1 can be deduced from the flat case (Theorem 4.1) by an appropriate lifting of the extension problem to the disk bundle $\mathscr{B}(\lambda)$ described in Section 2. We use the notation from that section.

To apply Theorem 4.1, use the defining function $\mathscr{T}$ given by (7). In locally trivial coordinates (x, s) on L^*_Z the function $\mathscr{T}$ is given by

$$\mathscr{T}(x, s) = T(x)s,$$

and then

$$d\mathscr{T}(x, s) = s dT(x) + T(x) ds.$$

In particular, the second term vanishes along $\mathcal{Z}$ while the first term vanishes along $\mathbb{O}_{L^*_Z}$.

Recall the Kähler metric

$$\tilde{\omega} := \frac{1}{c_o^{1/(n+1)}} \left(\pi^* \omega + c_o \sqrt{-1} \partial\bar{\partial} |v|^2 e^{\lambda} \right)$$

on $\mathscr{B}(\lambda)$, introduced in in Section 3.2 . There it was shown that

$$\text{Ricci}(\tilde{\omega}) = \pi^*(\text{Ricci}(\omega) - \partial\bar{\partial}\lambda).$$

Also defined in the same section was the line bundle

$$\Lambda_m := \pi^*(L \otimes L_Z^{*\otimes m})$$

with its singular Hermitian metric

$$e^{-\Psi} := \frac{e^{-\tilde{\varphi}+m\tilde{\lambda}}}{(|\sigma|^2 e^{\tilde{\lambda}})^{1-k-\delta_o}},$$

whose curvature is

$$\partial\bar{\partial}\Psi = \pi^*(\partial\bar{\partial}\varphi - (k+m-1+\delta_o)\partial\bar{\partial}\lambda) + [\mathbb{O}_{L_Z^*}].$$

In particular, given δ in Theorem 1, if m is the unique positive integer such that

$$\delta_o := \delta - m + 1 \in (0,1],$$

then under the hypotheses of Theorem 1

$$\partial\bar{\partial}\Psi + \mathrm{Ricci}(\tilde{\omega}) \geq \pi^*(\partial\bar{\partial}\varphi + \mathrm{Ricci}(\omega) - (k+1+\delta)\partial\bar{\partial}\lambda) \geq 0.$$

With this notation, Theorem 4.1 reads as follows.

THEOREM 5.1. *For every pair of $(k+1)$-tuples*

$$(f_o^{\mathcal{Z}}, ..., f_k^{\mathcal{Z}}) \in (\Gamma_{\mathcal{O}}(\mathcal{Z}, \Lambda_m))^{k+1} \quad \textit{and} \quad (f_o^{\mathbb{O}}, ..., f_k^{\mathbb{O}}) \in (\Gamma_{\mathcal{O}}(\mathbb{O}_{L_Z^*}, \Lambda_m)^{k+1},$$

associated to the pair of irreducible components $\mathcal{Z}$ and $\mathbb{O}_{L_Z^*}$ of $\{\mathscr{T}=0\}$, such that

$$\sum_{j=0}^{k} \int_{\mathcal{Z}} \frac{|f_j^{\mathcal{Z}}|^2 e^{-\Psi}}{\pi^*(|dT|^2_\omega e^{-\lambda})|\sigma|^2 e^{\tilde{\lambda}}} d\mathscr{A} + \sum_{j=0}^{k} \int_{\mathbb{O}_{L_Z^*}} \frac{|f_j^{\mathbb{O}}|^2 e^{-\Psi}}{\pi^*(|T|e^{-\lambda})^2 |d\sigma|^2_{\tilde{\omega}} e^{\tilde{\lambda}}} d\mathscr{A} < +\infty$$

there exists $\tilde{F} \in \Gamma_{\mathcal{O}}(\mathscr{B}(\lambda), \Lambda_m)$ such that

$$\tilde{F}_\perp^{(j)} = f_j^{\tilde{Z}}, \textit{ along } \tilde{Z}, \quad \tilde{F}_\perp^{(j)} = f_j^{\mathbb{O}} \textit{ along } \mathbb{O}_{L_Z^*}, \qquad 0 \leq j \leq k,$$

and

$$\int_{\mathscr{B}(\lambda)} |\tilde{F}|^2 e^{-\Psi} d\mathscr{V} \leq C^k \left(\sum_{j=0}^{k} \int_{\tilde{Z}} \frac{|f_j^{\tilde{Z}}|^2 e^{-\Psi}}{\pi^*(|dT|^2_\omega e^{-\lambda})|\sigma|^2 e^{\tilde{\lambda}}} d\mathscr{A} + \sum_{j=0}^{k} \int_{\mathbb{O}_{L_Z^*}} \frac{|f_j^{\mathbb{O}}|^2 e^{-\Psi}}{\pi^*(|T|e^{-\lambda})^2 |d\sigma|^2_{\tilde{\omega}} w^{\tilde{\lambda}}} d\mathscr{A} \right),$$

where C is the universal constant of Theorem 4.1.

5.2. **Conclusion of the proof of Theorem 1.** Given the data

$$(f_o, ..., f_k) \in \bigoplus_{j=o}^{k} \Gamma_{\mathcal{O}}(Z, L \otimes L_Z^{*\otimes j})$$

in the hypotheses of Theorem 1, we define the sections

$$f_j^{\mathcal{Z}} := \pi^* f_j \otimes \sigma^{m+j} \in \Gamma_{\mathcal{O}}(\mathcal{Z}, \Lambda_m) \quad \text{and} \quad f_j^{\mathbb{O}} = 0, \quad 0 \leq j \leq k.$$

We compute that

$$\begin{aligned}
\int_{\mathcal{Z}} \frac{|f_j^{\mathcal{Z}}|^2 e^{-\Psi}}{\pi^*(|dT|^2_\omega e^{-\lambda})|\sigma|^2 e^{\tilde{\lambda}}} d\mathscr{A} &= \int_{\mathcal{Z}} \frac{\pi^*(|f_j|^2 e^{-\varphi+j\lambda})(|\sigma|^2 e^{\tilde{\lambda}})^{m+k-j}}{(|\sigma|^2 e^{\tilde{\lambda}})^{2-\delta_o}\pi^*(|dT|^2_\omega e^{-\lambda})} d\mathscr{A} \\
&= \frac{\pi}{k-j+\delta} \int_Z \frac{|f_j|^2 e^{-\varphi+j\lambda}}{|dT|^2 e^{-\lambda}} dA_\omega \\
&\leq \frac{\pi}{\delta} \int_Z \frac{|f_j|^2 e^{-\varphi+j\lambda}}{|dT|^2 e^{-\lambda}} dA_\omega,
\end{aligned}$$

where the second equality follows from Fubini's Theorem.

Theorem 5.1 provides us with a section $G \in \Gamma_{\mathcal{O}}(\mathscr{B}(\lambda), \Lambda_m)$ such that

$$G^{(j)}_{\perp} = f^{\tilde{Z}}_j, \text{ along } \mathcal{Z}, \quad G^{(j)}_{\perp} = 0 \text{ along } \mathbb{O}_{L^*_Z}, \qquad 0 \le j \le k,$$

and

$$\int_{\mathscr{B}(\lambda)} |G|^2 e^{-\Psi} d\mathscr{V} \le \frac{2\pi C^k}{\delta} \sum_{j=0}^{k} \int_{\tilde{Z}} \frac{|f^{\tilde{Z}}_j|^2 e^{-\Psi}}{\pi^*(|dT|^2_\omega e^{-\lambda})|\sigma|^2 e^{\tilde{\lambda}}} d\mathscr{A},$$

Writing G in its homogeneous expansion according to Proposition 2.1, we have

$$G = \sum_{\ell=0}^{\infty} (\pi^* F_\ell)\sigma^{\otimes \ell},$$

where $F_\ell \in \Gamma_{\mathcal{O}}(X, L \otimes L_Z^{*\otimes \ell})$. Restriction to $\mathcal{Z}$ shows that

$$(F_m)^{(j)}_{\perp} = f_j \quad \text{and} \quad (F_\ell)^{(j)}_{\perp} \equiv 0 \quad \text{along } Z, \qquad 0 \le j \le k,\ \ell \ne m,$$

and thus $(\pi^* F_m) \otimes \sigma^{\otimes m}$ is also a k-jet extension of $(f^{\tilde{Z}}_o, ..., f^{\tilde{Z}}_k)$ along $\tilde{Z}$. We set

$$F = F_m.$$

We then calculate that

$$\int_{\mathscr{B}(\lambda)} |\pi^* F \otimes \sigma^{\otimes m}|^2 e^{-\Psi} d\mathscr{V} = \int_{\mathscr{B}(\lambda)} \frac{\pi^*(|F|^2 e^{-\varphi})(|\sigma|^2 e^{\tilde{\lambda}})^{m+k}}{(|\sigma|^2 e^{\tilde{\lambda}})^{1-\delta_o}} d\mathscr{V} = \frac{2\pi}{k+1+\delta} \int_Z |F|^2 e^{-\varphi} dV_\omega,$$

Since the measure $e^{-\Psi} d\mathscr{V}$ is invariant with respect to multiplication by a unimodular constant in the fibers of L^*_Z, orthogonality implies that

$$\int_{\mathscr{B}(\lambda)} |\pi^* F \otimes \sigma^{\otimes m}|^2 e^{-\Psi} d\mathscr{V} \le \int_{\mathscr{B}(\lambda)} |G|^2 e^{-\Psi} d\mathscr{V}.$$

Thus

$$\int_Z |F|^2 e^{-\varphi} dV_\omega \le \frac{k+1+\delta}{\delta} C^k \sum_{j=0}^{k} \int_Z \frac{|f_j|^2 e^{-\varphi + j\lambda}}{|dT|^2 e^{-\lambda}} dA_\omega,$$

and the proof of Theorem 1 is complete. □

6. The proof of Theorem 2

Let $e^{-\varphi}$ be the singular Hermitian metric to be extended. Fix a sufficiently ample line bundle $A \to X$ and a smooth metric $e^{-\chi}$ with sufficiently large curvature form, i.e., $\sqrt{-1}\partial\bar{\partial}\chi >> \omega$. Since H is pseudoeffective, the line bundle

$$H_m := H^{\otimes m} \otimes A$$

is big, as it admits the singular Hermitian metric $e^{-(m\varphi_o + \chi)}$, with curvature current

$$\partial\bar{\partial}(m\varphi_o + \chi) \ge \partial\bar{\partial}\chi.$$

Thus replacing A by $A^{\otimes 2}$ if necessary, we may assume H_m is ample.

Consider the metric $e^{-(m\varphi+\chi)}$ for $H_m|_Z$. If χ is sufficiently positively curved, the hypotheses of Lemma 2.9 are satisfied. With the notation of the latter lemma, choose an orthonormal basis $f_{1,(m)},...,f_{N_m,(m)}$ for $\mathscr{H}^2_m(\varphi)$ and denote by $e^{-\theta_m}$ the metric for $H|_Z$ defined by the potential

$$\theta_m := -\frac{\chi}{m} + \frac{1}{m}\log\sum_{j=1}^{N_m}|f_j^{(m)}|^2.$$

Since $e^{-\varphi_o}|_Z \le e^{-\varphi}$,

$$\int_Z |f_j^{(m)}|^2 e^{-m\varphi_o-\chi}dV_\omega \le \int_Z |f_j^{(m)}|^2 e^{-m\varphi-\chi}dV_\omega = 1. \tag{12}$$

Now denote by $\gamma_{j,(m)}$ the normal m-jets along Z associated to the $(m+1)$-tuple

$$(f_{1,(m)},0,0,...,0) \in \bigoplus_{\ell=0}^{m}\Gamma_{\mathcal{O}}(Z, H_m\otimes L_Z^{*\otimes\ell}),$$

for all indices j with $1\le j\le N_m$. Theorem 1 yields sections $F_{j,(m)}\in H^0(X,H_m)$, $1\le j\le N_m$, such that

$$J^m_\perp F_{j,(m)} = \gamma_{j,(m)} \quad \text{and} \quad \int_X |F_{j,(m)}|^2 e^{-m\varphi_o-\chi}dV_\omega \le C_o^m, \tag{13}$$

assuming, again, that $\sqrt{-1}\partial\bar\partial\chi$ is sufficiently large. Define the metric $e^{-\Theta_m}$ for $H\to X$ whose potential is

$$\Theta_m = -\frac{\chi}{m} + \frac{1}{m}\log\sum_{j=1}^{N_m}|F_{j,(m)}|^2.$$

Since, by definition, $\Theta_m|_Z = \theta_m$, Lemma 2.9 provides the upper estimate

$$e^{-\Theta_m}|_Z \le C^{1/m}e^{-\varphi}. \tag{14}$$

This upper bound, which uses the L^2 extension theorem for sections, does not require Theorem 1 on L^2 extension for jets; the estimate (13) will be needed below.

Let us turn to the upper bound. Consider the section $F_{j,(m)}$. To analyze the section, let us work in a neighborhood $U_j = V_j\cap\mathbb{D}(z,r)$ for sufficiently small r. Choose a frame for L_Z and a holomorphic coordinate t in the normal direction to Z such that $U_j = V_j\times\{|t|<2\delta\}$ for some small number δ. Write (z,t) for the coordinates in $V_j\times\{|t|<2\delta\}$. Expanding the section in a Taylor series in t yields

$$F_{i,(m)}(z,t) = f_{i,(m)}(z) + \sum_{k=m+1}^{\infty} c_{i,k}(z)t^k,$$

where $c_{i,k}\in\mathcal{O}(V_j)$. (In view of (13), $c_{i,k}(z)\equiv 0$ for $1\le k\le m$.) By the Cauchy formula,

$$c_{i,k}(z) = \frac{1}{2\pi\sqrt{-1}}\int_{|t|=\delta} F_{i,(m)}(z,t)\frac{dt}{t^{k+1}}.$$

Thus by the estimate (13) (and this is where the theorem on jet extensions is crucially needed)

$$|c_{i,k}(z)| \le \frac{1}{2\pi}\int_0^{2\pi}\delta^{-k}|F_{i,(m)}(z,\delta e^{\sqrt{-1}\theta})|d\theta \le C_1^m\delta^{-k}.$$

Both inequalities follow from the sub-mean value property, and in the second inequality we also used the estimate $e^{m\varphi+\chi} = (e^{\varphi})^m e^{\chi} \le C_2^m$, which follows from the compactness of X. It follows that for $|t| \le \delta \le 1/2$,

$$|F_{i,(m)}(z,t) - f_{i,(m)}(z)| \le \sum_{k=m+1}^{\infty} C^m \delta^k = (C\delta)^m \frac{\delta}{1-\delta} \le (C\delta)^m. \tag{15}$$

Returning to the proof of Theorem 2, note that for $|t| \le \delta$ one has the estimate

$$\begin{aligned}
\frac{1}{2}\Theta_m(z,t) &= \frac{-\chi(z,t)}{2m} + \frac{1}{m}\log\left(\sum_{j=1}^{N_m} |F_{j,(m)}(z,t)|^2\right)^{1/2} \\
&\le \frac{-\chi(z,t)}{2m} + \frac{1}{m}\log\left(\left(\sum_{j=1}^{N_m} |f_{j,(m)}(z)|^2\right)^{1/2}\right. \\
&\qquad \left. + \left(\sum_{j=1}^{N_m} \left|F_{j,(m)}(z,t) - f_{j,(m)}(z)\right|^2\right)^{1/2}\right) \\
&\le \frac{-\chi(z,t)}{2m} + \frac{1}{2m}\log\left(\sum_{j=1}^{N_m} |f_{j,(m)}(z)|^2\right) \\
&\qquad + \left(\sum_{j=1}^{N_m} \left|F_{j,(m)}(z,t) - f_{j,(m)}(z)\right|^2\right)^{1/m}.
\end{aligned}$$

The last inequality follows from the basic calculus estimate $\frac{2}{m}\log(a+b) \le b^{2/m} + \frac{2}{m}\log a$, which holds for any $a > 0$, $m > 0$ and $0 \le b \le 1$. In view of (15), when $|t| \le \delta^{m+1}$,

$$\Theta_m(z,t) \le \frac{-\chi(z,t)}{m} + \frac{1}{m}\log\left(\sum_{j=1}^{N_m} |f_{j,(m)}(z)|^2\right) + (N_m)^{1/m} C\delta.$$

Since $\mathscr{H}^2_m(\varphi) \subset H^0(Z, H_m)$, $N_m = O(m^{n-1})$ by asymptotic Riemann-Roch, and thus for sufficiently large m, $(N_m)^{1/m} \le 2$. Finally, taking $\delta = \frac{\varepsilon}{2C}$ where $\varepsilon > 0$ is a fixed number, we obtain from (12) and Lemma 2.9 the estimate

$$\Theta_m(z,t) \le \frac{-\chi(z,t)}{m} + \sup_{B_{\frac{\varepsilon}{2C}}(z,t)} \left(\varphi + \frac{\chi}{m}\right) + \frac{C_1 \log m}{m} + \varepsilon. \tag{16}$$

We are now ready to complete the proof of Theorem 2. Define the metric $e^{-\Phi}$ for $H \to X$ by setting

$$\Phi := \limsup_{y \to x} \limsup_{m \to \infty} \Theta_m.$$

Then evidently $\sqrt{-1}\partial\bar{\partial}\Phi \ge 0$ and Φ is locally upper semi-contiuous. Then (14) implies

$$e^{-\Phi}|_Z \le e^{-\varphi},$$

while (16) shows that

$$e^{-\Phi}|_Z \ge e^{-\varphi}.$$

The proof of Theorem 2 is therefore complete. □

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