Bloch Constants in One and Several Variables

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ABSTRACT: We give covering theorems in one variable for holomorphic functions on the unit disc with k-fold symmetry. In the case of convex maps we give a generalization, shown to us by D. Minda, to the case where $a_2 = \ldots = a_k = 0$. In several variables we determine the Bloch constant (equivalently the Koebe constant) for convex maps of B_n with k-fold symmetry, $k \geq 2$. We also estimate and in some cases compute the Bloch constant for starlike maps of B_n with k-fold symmetry. We compare the Bloch constant with the Koebe constant for such maps and determine values of n and k for which equality holds.

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1. INTRODUCTION

This paper is the result of an attempt to answer the following

Question: If $F : B_n \to \mathbf{C}^n$ is a biholomorphic map of the unit ball onto a convex domain such that $dF(0) = I$, must $F(B_n)$ contain a ball of radius $\pi/4$?

This is the problem of the univalent Bloch constant for convex maps in several variables. When $n = 1$ the result is of course true [Sz,Z,M2]. There are at least three different ways of proving this – a classical proof due to Szegö [Sz], a differential-geometric proof based on ultrahyperbolic metrics [Z,M2], and a proof based on the growth estimate $|f(z)| \geq \arctan|z|$ for convex functions with vanishing second coefficient, cf. [Gro,J,F]. (Concerning the third proof, compare the recent work of Bonk [Bo] and the generalization of Bonk's distortion theorem to several variables by Liu [Li].)

Some evidence that the answer to the above question in several variables is also affirmative is contained in our paper [GV], where the following was proved: if $F : B_n \to \mathbf{C}^n$ is a holomorphic map with convex image and $dF(0) = I$, then any translate of $F(B_n)$ through a distance less than $\pi/2$ has nonempty intersection with $F(B_n)$. This is the several variables analog of a theorem of MacGregor [McG], which of course does not require convex image in one variable.

In this paper we answer the question in the affirmative if f is odd or has k-fold symmetry (if $k > 2$ larger constants are obtained), though we are still unable to settle the general case. Our results go beyond the case of convex maps, however. We begin in Section 2 with covering theorems for functions on the unit disc with k-fold symmetry, or, more generally in the convex case, functions with $a_2 = \ldots = a_k = 0$. (The latter argument was shown to us by David Minda.) In Section 3 we obtain analogs of the results with k-fold symmetry for maps from the unit disc into $\mathbf{C}^n$. Our version of the Bloch or Koebe theorem for convex maps of the unit ball in $\mathbf{C}^n$ with k-fold symmetry appears in Section 4. It is based on a projection argument and a one-variable covering theorem for the convex hull of the image of a k-fold symmetric function. Some rather unexpected results for starlike maps of the ball are obtained in Section 5. The Koebe constant for such maps, both in the general case and in the presence of k-fold symmetry, was determined by Barnard, FitzGerald, and Gong [BFG]. For general starlike maps and also in the case of odd starlike maps we give upper bounds for the Bloch constant which decrease as the dimension increases and approach the Koebe constant as $n \to \infty$. In the k-fold symmetric case ($k \geq 3$) a new phenomenon occurs: the Bloch constant coincides with the Koebe constant except when $k = 3$ and $n = 2, 3$. Finally we have one result for the polydisc (Section 6) - the Bloch constant for all convex univalent maps can be computed rather easily using a characterization of such maps due to Suffridge [Su1].

2. ONE-VARIABLE RESULTS

In this section we give covering theorems of Bloch or Koebe type for functions with k-fold symmetry, i.e. functions such that $e^{-2\pi i/k} f(e^{2\pi i/k} z) = f(z)$ where k is a positive integer. In the convex case we give a generalization, shown to us by David Minda, to the case where $a_2 = a_3 = \cdots = a_k = 0$. We denote by $\Delta(p, r)$ the open disc of centre p and radius r. The unit disc is denoted by Δ.

Theorem 1.

(i) *Suppose $f : \Delta \to \mathbf{C}$ is univalent with k-fold symmetry and normalized by $f'(0) = 1$. Then $f(\Delta) \supseteq \Delta(0, 4^{-1/k})$.*

(ii) *Suppose further that f is convex. Then*

$$f(\Delta) \supseteq \Delta(0, r_k) \quad \text{where} \quad r_k = \int_0^1 \frac{dt}{(1+t^k)^{2/k}}.$$

Both results are sharp.

Proof: Part (i)follows from the growth estimate

$$|f(z)| \geq \frac{|z|}{(1+|z|^k)^{2/k}} \tag{2.1}$$

obtained by considering the k-th root transform of f [Neh, pp. 225-226, problem 13]. Part (ii) is obtained by applying (2.1) to the function $h(z) = zf'(z)$ and then integrating.

For the first assertion, we see that the results are sharp for each k by considering the functions

$$f_k(z) \;=\; (K(z^k))^{1/k}, \quad \text{where} \quad K(z) \;=\; \frac{z}{(1-z)^2}$$

is the Koebe function. For the case of convex functions, we see that the functions

$$g_k(z) \;=\; \begin{cases} \frac{z}{1-z} & k=1 \\ \frac{1}{2}\log\frac{1+z}{1-z} & k=2 \\ \int_0^z \frac{d\tau}{(1-\tau^k)^{2/k}} & k\geq 3 \end{cases} \tag{2.2}$$

gives sharp results. For $k \geq 3$ the map is a (normalized) Riemann map onto a regular polygon of order k [Neh, p. 196, problem 4]. Note that in all cases we have $zg_k'(z) = f_k(z)$.

Now we come to Minda's generalization. First we recall the

Generalized Schwarz Lemma: Suppose $\phi : \Delta \to \Delta$ is holomorphic, and $\phi(0) = \phi'(0) = \cdots = \phi^{(k-1)}(0) = 0$. Then

$$(i) \qquad \frac{1}{k!}|\phi^{(k)}(0)| \;\leq\; 1$$

$$(ii) \qquad |\phi(z)| \;\leq\; |z|^k \quad z \in \Delta$$

Equality holds in (i) or (at a single point) in (ii) iff $\phi(z) = \lambda z^k$ where λ is a unimodular constant.

Using this we obtain

Theorem 2. *Suppose $\Omega \subset \mathbf{C}$ is a domain, $F : \Delta \to \Omega$ is a universal covering, $f : \Delta \to \Omega$ is holomorphic, and $f(0) = F(0)$. If $f'(0) = \cdots = f^{(k-1)}(0) = 0$ then*

$$(i) \qquad \frac{1}{k!}|f^{(k)}(0)| \;\leq\; |F'(0)|$$

$$(ii) \qquad f(\{z : |z| \leq r\}) \;\subseteq\; F(\{z : |z| \;\leq\; r^k\}) \quad 0 < r < 1.$$

Equality holds in (i) if and only if $f(z) = F(\lambda z^k)$ for some unimodular constant λ. Also if $f(z) \in \partial F(\{|z| \leq r^k\})$ for some z such that $|z| = r$ then $f(z) = F(\lambda z^k)$ for some unimodular constant λ.

Proof: Since F is a covering, there is a holomorphic function $\phi : \Delta \to \Delta$ with $\phi(0) = 0$ such that $F \circ \phi = f$. The result now follows from the Generalized Schwarz Lemma.

Remark: The conclusion of the theorem is what would follow from $f(z) \prec F(z^k)$, but this subordination is not generally valid.

Theorem 3. *Suppose that $f(z) = z + a_{k+1}z^{k+1} + a_{k+2}z^{k+2} + \cdots$ is convex univalent in Δ. Then for $z \in \Delta$*

$$\frac{1}{(1+|z|^k)^{2/k}} \ \le\ |f'(z)| \ \le\ \frac{1}{(1-|z|^k)^{2/k}}.$$

These bounds are sharp and are realized by the functions g_k in (2.2).

Proof. Since f is convex univalent, we have

$$Re\left(\frac{zf''(z)}{f'(z)}\right) \ >\ -1, \quad z \in \Delta.$$

Hence the function $g(z) = \frac{zf''(z)}{f'(z)}$ is subordinate to $G(z) = \frac{1+z}{1-z} - 1 \ =\ \frac{2z}{1-z}$. Now $g(0) = 0$ and

$$g(z) \ =\ \frac{(k+1)kz^k + \cdots}{1 + (k+1)a_{k+1}z^k + \cdots}$$

so $g'(0) = \cdots = g^{(k-1)}(0) = 0$. We conclude from Theorem 2 that $g(\{|z| \le r\}) \le G(\{|z| \le r^k\})$. The latter set is a disc centered on the real axis. Since

$$G(r^k) \ =\ \frac{2r^k}{1-r^k} \quad \text{and} \quad G(-r^k) \ =\ \frac{-2r^k}{1+r^k},$$

the centre of this disc is located at $\frac{2r^{2k}}{1-r^{2k}}$ and its radius is $\frac{2r^k}{1-r^{2k}}$. Thus for $z \in \Delta$,

$$\left|g(z) - \frac{2|z|^{2k}}{1-|z|^{2k}}\right| \ \le\ \frac{2|z|^k}{1-|z|^{2k}}$$

which gives in turn

$$\left|\frac{zf''(z)}{f'(z)} - \frac{2|z|^{2k}}{1-|z|^{2k}}\right| \le \frac{2|z|^k}{1-|z|^{2k}}$$

$$\left|\frac{|z|^2f''(z)}{f'(z)} - \frac{2\overline{z}|z|^{2k}}{1-|z|^{2k}}\right| \le \frac{2|z|^{k+1}}{1-|z|^{2k}}$$

$$\left|\frac{f''(z)}{f'(z)} - \frac{2\overline{z}|z|^{2k-2}}{1-|z|^{2k}}\right| \le \frac{2|z|^{k-1}}{1-|z|^{2k}}.$$

It suffices to establish the result for $z = r > 0$. In this case we have

$$\left| \frac{f''(r)}{f'(r)} - \frac{2r^{2k-1}}{1-r^{2k}} \right| \;\le\; \frac{2r^{k-1}}{1-r^{2k}}.$$

By integration,

$$\begin{aligned} \left| \log f'(r) + \frac{1}{k}\log(1-r^{2k}) \right| \;&\le\; \int_0^r \frac{2\rho^{k-1}}{1-\rho^{2k}} d\rho \\ &= \frac{1}{k}\log\frac{1+r^k}{1-r^k}. \end{aligned}$$

Since $|w| \le R$ implies $-R \le Rew \le R$, we obtain

$$-\frac{1}{k}\log\frac{1+r^k}{1-r^k} \;\le\; \log|f'(r)| + \frac{1}{k}\log(1-r^{2k}) \;\le\; \frac{1}{k}\log\frac{1+r^k}{1-r^k}$$

or

$$\frac{1}{k}\log\frac{1}{(1+r^k)^2} \;\le\; \log|f'(r)| \;\le\; \frac{1}{k}\log\frac{1}{(1-r^k)^2}$$

or finally

$$\frac{1}{(1+r^k)^{2/k}} \;\le\; |f'(r)| \;\le\; \frac{1}{(1-r^k)^{2/k}}.$$

Corollary. *For the class of univalent convex functions of the form $f(z) = z + a_{k+1}z^{k+1} + a_{k+2}z^{k+2} + \cdots$, the Bloch constant coincides with the Koebe constant and has the value $r_k = \int_0^1 \frac{dt}{(1+t^k)^{2/k}}$. (In the absence of univalence but with the assumption of k-fold symmetry the Landau constant has this value.)*

Next we give a covering theorem for the convex hull of the image of a map with k-fold symmetry. This result does not require univalence; however it does not generalize to the case of maps with $a_2 = \cdots = a_k = 0$. (Because of this our covering theorem for convex maps of the unit ball in $\mathbf{C}^n$ (Theorem 7) requires a symmetry assumption.)

Theorem 4. *Let $f : \Delta \to \mathbf{C}$ have k-fold symmetry and satisfy $f'(0) = 1$. Then $\widehat{f(\Delta)} \supseteq \Delta(0, r_k)$ where $\widehat{\ }$ denotes the convex hull.*

Proof: (For the case $k = 1$, the theorem is well-known.) We may assume $\widehat{f(\Delta)} \ne \mathbf{C}$. It is clear that $K = \widehat{f(\Delta)}$ has k-fold symmetry. Let $\omega \in \partial K$ be a point of least modulus.

Then the supporting line at ω must be perpendicular to $\overline{O\omega}$. (Otherwise we obtain a contradiction to the choice of ω.) Let $M = |\omega|$. Then f is subordinate to the function $h(z) = \frac{M}{r_k} g_k(z)$ with g_k as given in (2.2). Thus $h'(0) = M/r_k \geq f'(0) = 1$, so that $M \geq r_k$. The proof is complete.

3. RELATED RESULTS FOR MAPS FROM THE UNIT DISC INTO $\mathbf{C}^n$.

The proof of Theorem 4 can be combined with the Hahn-Banach theorem to show

Theorem 5. *Let $f : \Delta \to \mathbf{C}^n$ be a holomorphic map with k-fold symmetry. (Thus $f(0) = 0$.) Then the closed convex hull of $f(\Delta)$ contains the disc $\{zf'(0) |\ |z| \leq r_k\}$.*

The details are omitted (cf. [Gra2, Lemma 4]). This result allows one to improve the constant in estimates for the Kobayashi metric on convex domains with k-fold symmetry when the base point is the origin (cf. [Gra1, Gra2]). We recall that the Kobayashi metric is the nonnegative function on the tangent bundle to a domain Ω defined by

$$K(p;\xi) = \inf\{|v| \ : \ v \in T_0(\Delta) \quad \text{and} \quad \exists f : \ \Delta \to \Omega \quad \text{holomorphic}$$
$$\text{such that} \quad f(0) = p \text{ and } df_0(v) = \xi\}.$$

Theorem 6. *Let Ω be a convex domain in $\mathbf{C}^n$ with k-fold symmetry. Let $\xi \in T_0\Omega$ and let $r(0;\xi)$ denote the radius of the largest disc centered at 0, tangent to ξ, and contained in Ω. Then*

$$r_k \frac{|\xi|}{r(0;\xi)} \ \leq \ K(0;\xi) \ \leq \ \frac{|\xi|}{r(0,\xi)}.$$

Proof: The upper estimate is well-known. To prove the lower one, note that if $f : \Delta \to \Omega$ is holomorphic and $f(0) = 0$, then the k-fold symmetric map

$$g(z) \ = \ \frac{1}{k} \sum_{j=1}^{k} e^{-2\pi ij/k} f(e^{2\pi ij/k} z)$$

satisfies $g(\Delta) \ \subset \ \Omega, \quad g(0) = 0$, and $g'(0) = f'(0)$. We now apply Theorem 5 and note that the disc in the conclusion of the theorem must be contained in $\overline{\Omega}$.

4. THE BLOCH, LANDAU, AND KOEBE CONSTANTS FOR CONVEX MAPS OF B_n WITH k-FOLD SYMMETRY.

In this section we determine the Bloch (Landau) constant for normalized univalent convex maps of B_n with k-fold symmetry, $k \geq 2$. It coincides with the Koebe constant for the same class of maps. (In fact one can relax the requirement of univalence here if one defines the Koebe constant to be the radius of the largest ball in the image of the map centered at $F(0)$.) That is, in the presence of the symmetry condition $e^{-2\pi i/k}F(e^{2\pi i/k}\zeta) = F(\zeta)$, $\zeta \in B_n$, it suffices to consider balls in $F(B_n)$ which are centered at 0.

We first formulate a Koebe theorem for convex maps which satisfy a weaker symmetry condition - a k-fold symmetry condition for a particular slice only.

Definition: Let $F : B_n \to \mathbf{C}^n$ be a holomorphic map such that $F(B_n)$ is convex, $F(0) = 0$, and $dF(0) = I$. F is said to have critical - slice symmetry of order k if there is a point $\omega \in \partial F(B_n)$ at minimum distance to 0 such that, on setting $a = \omega/||\omega||$, the function $\phi(z) =< F(za), a >$ has symmetry of order k.

Theorem 7. *Let $F : B_n \to \mathbf{C}^n$ have critical-slice symmetry of order k. Then $F(B_n) \supseteq B_n(0, r_k)$ where $r_k = \int_0^1 \frac{dt}{(1+t^k)^{2/k}}$. This result is sharp.*

Proof: With notation as in the definition, let π_a be the orthogonal projection of $\mathbf{C}^n$ onto the plane $\mathbf{C}a$. By Theorem 4, $\widehat{\phi(\Delta)} \supseteq \Delta(0, r_k)$. But $\widehat{\phi(\Delta)} \subseteq \pi_a(F(B_n))$, so $\pi_a(F(B_n)) \supseteq \Delta(0, r_k)$. Let M be the (unique) supporting hyperplane at ω. Then $a \perp M$. Thus $\pi_a(M)$ is a line, and since $\pi_a\omega = \omega$, we have $|\omega| \geq r_k$.

The sharpness of the constants follows from the following beautiful result of Roper and Suffridge [Ro,RS]: if $f : \Delta \to \mathbf{C}$ is a convex univalent function such that $f'(0) = 1$, then there exists a convex univalent map $F : B_n \to \mathbf{C}^n$ normalized by $dF(0) = I$ such that $F(z_1, 0, \ldots, 0) = (f(z_1), 0, \ldots, 0)$. (Also if f is k-fold symmetric then so is F.)

Corollary. *The Bloch, Landau, and Koebe constants for normalized univalent convex maps of B_n with k-fold symmetry all coincide and have the value $r_k (k \geq 2)$. In the case* *of the Koebe and Landau constants one can relax the requirement of univalence.*

Proof: All that remains to complete the proof is to observe that if $\Omega \subset \mathbf{C}^n$ is a convex domain with k-fold symmetry and $B \subset \Omega$ is a ball, then the convex hull of the balls $e^{2\pi ij/k}B,\ j = 1, \ldots, k$ contains a ball centered at 0 which is at least as large as B.

5. STARLIKE MAPS OF THE UNIT BALL IN $\mathbf{C}^n$.

As in one variable a biholomorphic map $F : B_n \to \mathbf{C}^n$ such that $F(0) = 0$ is said to be starlike if whenever $w \in F(B_n)$ then $tw \in F(B_n)$ for $0 \le t \le 1$. It is customary to normalize such maps by requiring that $dF(0) = I$.

We give an upper bound on the Bloch constant for starlike maps of $B_n, n \ge 2$ which decreases with the dimension and which tends to $\frac{1}{4}$ as $n \to \infty$. Of course $\frac{1}{4}$ is the value of the Koebe constant for starlike maps of the ball [BFG]. We also consider starlike maps with k-fold symmetry. In the odd case, again an upper bound is given which decreases with the dimension and tends to the Koebe constant for such maps. However in the presence of higher order symmetry, a new phenomenon appears: the Bloch constant coincides with the Koebe constant except when $k = 3$ and $n = 2$ or 3. This behaviour does not take place on the polydisc.

Theorem 8. *The Bloch constant b_n for starlike maps of the unit ball in $\mathbf{C}^n (n \ge 2)$ satisfies $b_n < \frac{1}{4}\left(\frac{\sqrt{n}+1}{\sqrt{n}-1}\right)^2$.*

Proof: Let $K(z) = z/(1-z)^2$ denote the Koebe function and consider the map $\zeta \mapsto (K(\zeta_1), \ldots, K(\zeta_n))$ in n variables. This map is starlike [BFG] and omits the hyperplanes $w_j = -\frac{1}{4},\ j = 1, \ldots, n$. It is naturally defined on the unit polydisc but we are considering the restriction to the unit ball. If there is a ball of radius r in the image of this map, its centre must be a distance at least r from each of these hyperplanes. Let $(c_1, \ldots, c_n)$ denote the coordinates of the centre of such a ball. The representation $K(z) = -\frac{1}{4} + \frac{1}{4}\left(\frac{1+z}{1-z}\right)^2$ shows that for a given value of $|z|,\ |K(z) + \frac{1}{4}|$ is maximized when z is positive real.

Conversely for a given value of $|K(z)+\frac{1}{4}|$ (larger than $\frac{1}{4}$), $|z|$ is minimized when z is positive real. Hence for the purposes of bounding r above we may assume that $c_1 = \cdots = c_n = r - \frac{1}{4}$. Solving $\frac{x}{(1-x)^2} = r - \frac{1}{4}$ and requiring that $x < n^{-\frac{1}{2}}$ gives $r < \frac{1}{4}\left(\frac{\sqrt{n}+1}{\sqrt{n}-1}\right)^2$.

Theorem 9. *The Bloch constant $b_n^{(2)}$ for odd starlike maps of B_n satisfies*

$$b_n^{(2)} \;<\; \frac{1}{2}\frac{n+1}{n-1}.$$

Proof: Consider the map $\zeta \mapsto \left(\frac{\zeta_1}{1-\zeta_1^2}, \ldots, \frac{\zeta_n}{1-\zeta_n^2}\right)$, i.e. in each variable we have the square root transform of the Koebe function. Now for the map $z \mapsto \frac{z}{1-z^2}$, if we fix $|z|$ then the modulus of the image point and (what we really want) its distance from the omitted rays are maximized when z is real (say positive). The image of the above n-variable map omits the hyperplanes $w_j = \pm i/2, \;\; j = 1, \ldots, n$. Suppose there is a ball of radius $r > \frac{1}{2}$ in the image of this map. For the purposes of bounding r we may assume that the centre of this ball is at $(c, c, \ldots, c)$ where $c > 0$. Then $r \leq \sqrt{c^2 + \frac{1}{4}}$. If we set $c = \frac{x}{1-x^2}$ and require $0 < x < n^{-\frac{1}{2}}$ we obtain $r < \frac{1}{2}\frac{n+1}{n-1}$.

Theorem 10. *The Bloch constant for starlike maps of the ball in $\mathbf{C}^n$ with k-fold symmetry is given by $b_n^{(k)} = 4^{-1/k}$ when $k = 3$ and $n \geq 4$ and when $k \geq 4$ and $n \geq 2$. (This is the value of the Koebe constant for such maps [BFG].)*

Proof: We consider the map $\zeta \mapsto F_k(\zeta) = \left(\frac{\zeta_1}{(1-\zeta_1^k)^{2/k}}, \ldots, \frac{\zeta_n}{(1-\zeta_n^k)^{2/k}}\right)$. This is a starlike map which covers the ball of radius $r^{-1/k}$ centered at 0 in $\mathbf{C}^n$ [BFG]. The one-variable map $z \mapsto f_k(z) = \frac{z}{(1-z^k)^{2/k}}$ covers a disc of the same radius centered at 0. However for $k \geq 3, \quad r(f_k, w)$ has a local maximum at 0.

We now treat the cases $k = 3$ and $k > 3$ separately. When $k = 3$ the endpoints of the three rays omitted by f_3 are located at the points $4^{-\frac{1}{3}}e^{2\pi i\ell/3}, \; \ell = 1, 2, 3$, and we must move a distance $4^{-\frac{1}{3}}$ from 0 to find a nonzero w such that $r(f_3, w)$ is as large as $4^{-\frac{1}{3}}$. To minimize $|z|$ such that $f_k(z)$ is at distance $4^{-\frac{1}{3}}$ from 0 we take z^3 to be positive, hence we may take $x > 0$. Solving for x from $\frac{x}{(1-x^3)^{2/3}} = 4^{-\frac{1}{3}}$ gives $x = \left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)^{\frac{1}{3}}$. Requiring that $x < n^{-\frac{1}{2}}$ gives $n < \left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)^{\frac{2}{3}} \simeq 3.24$. Hence only $n = 2, 3$ are possible.

When $k > 3$ the entire omitted rays of f_k come into play rather than just the endpoints. A point on the positive real axis which is at distance $a \geq 4^{-\frac{1}{k}}$ from the nearest omitted ray must have distance $a\ csc(\pi/k)$ from 0. Solving $f_k(x) = 4^{-\frac{1}{k}} csc(\pi/k)$ and requiring that $0 < x < n^{-\frac{1}{2}}$ gives

$$n \ < \ \left(\frac{\sqrt{1 + (csc(\pi/k))^k} + 1}{\sqrt{1 + (csc(\pi/k))^k} - 1} \right)^{2/k} .$$

The right-hand side has the approximate value 1.6 when $k = 4$ and is clearly a decreasing function of k. Hence there are no values of $n \geq 2$ which satisfy this.

Remark. For the cases $k = 3,\ n = 2, 3$ not covered by Theorem 9 we can estimate the Bloch constant by the method of Theorem 7 and 8. This gives $b_2^{(3)} < 0.8340...$ and $b_3^{(3)} < 0.6486...$.

6. THE BLOCH CONSTANT FOR CONVEX MAPS OF THE POLYDISC.

The Bloch constant can be determined precisely for univalent convex maps of the polydisc Δ^n using a characterization of such maps due to Suffridge [Su1, Theorem 3]. We have

Theorem 11. *Let $F : \Delta^n \to \mathbf{C}^n$ be a univalent map with convex image normalized by $dF(0) = I$. Then $F(B_n)$ contains a polydisc each of whose radii is $\pi/4$. (Hence it contains a ball of this radius.) This result is sharp.*

Proof: According to Suffridge's characterization F has the representation

$$F(\zeta) = T(f_1(\zeta_1), f_2(\zeta_2), \ldots, f_n(\zeta_n))$$

where T is a non-singular linear transformation and the f_j are univalent convex functions of one variable. Since $dF(0) = I$, T must be a diagonal matrix, and after absorbing constants into the f_j we may assume that $T = I$ and $f_j'(0) = 1,\ j = 1, \ldots, n$. Now the image of each f_j contains a disc of radius $\pi/4$ so the image of F must contain a polydisc of the type described. Sharpness follows by considering an n-tuple of functions of the form $\frac{1}{2}\log\frac{1+z}{1-z}$.

REFERENCES

[A] L.V. Ahlfors, An extension of Schwarz's lemma. Trans. Amer. Math. Soc. **43** (1938), 359-364.

[BFG] R.W. Barnard, C.H. FitzGerald, and S. Gong, The growth and $\frac{1}{4}$-theorems for starlike mappings in $\mathbf{C}^n$. Pacific J. Math **150** (1991), 13-22.

[Bo] M. Bonk, On Bloch's constant. Proc. Amer. Math. Soc. **110** (1990), 889-894.

[Bl] A. Bloch, Les théorèmes de M. Valiron sur les fonctions entières et la théorie de l'uniformization, Ann. Fac. Sci. Univ. Toulouse **17** (1925), 1-22.

[F] M. Finkelstein, Growth estimates of convex functions. Proc. Amer. Math. Soc. **18** (1967), 412-418.

[Go] G.M. Golusin, Geometric Theory of Functions of a Complex Variable. Translations of Mathematical Monographs, Vol. 26, American Mathematical Society, Providence, R.I. 1969.

[Gra1] I. Graham, Distortion theorems for holomorphic maps between convex domains in $\mathbf{C}^n$, Complex Variables **15** (1990), 37-42.

[Gra2] _______, Sharp constants for the Koebe theorem and for estimates of intrinsic metrics on convex domains. Proc. Symp. Pure Math. **52** (1991), Part 2, 233-238.

[GV] _______ and D. Varolin, On translations of the images of analytic maps. To appear in Complex Variables.

[Gro] T.H. Gronwall, On the distortion in conformal mapping when the second coefficient in the mapping function has an assigned value. Proc. Nat. Acad. Sci. U.S.A. **6** (1920), 300-302.

[J1] J.A. Jenkins, On a problem of Gronwall, Ann. Math. **59** (1954), 490-504.

[La] E. Landau, Über die Blochsche Konstante und zwei verwandte Weltkonstanten, Math. Z. **30** (1929), 608-634.

[Li] X. Liu, Bloch functions of several complex variables, Pacific J. Math. **152** (1992), 347-363.

[LM] _______ and D. Minda, Distortion theorems for Bloch functions, Trans. Amer. Math. Soc. **333** (1992), 325-338.

[McG] T.H. MacGregor, Translations of the image domains of analytic functions, Proc. Amer. Math. Soc. **16** (1965), 1280-1286.

[M1] D. Minda, Bloch constants, J. Analyse Math. **41** (1982), 54-84.

[M2] _______, Lower bounds for the hyperbolic metric in convex regions, Rocky Mtn. J. Math. **13** (1983), 61-69.

[M3] _______, The Bloch and Marden constants, Computational Methods and Function Theory (Valparaiso 1989), 131-142, Lecture Notes in Mathematics 1435, Springer Verlag, Berlin-Heidelberg- New York 1990.

[Neh] Z. Nehari, Conformal Mapping, McGraw-Hill, New York 1952.

[Ro] Kevin Roper, Thesis, University of Kentucky, to appear.

[RS] K. Roper and T. Suffridge, Convex Mappings on the Unit Ball of $\mathbf{C}^n$. Preprint, University of Kentucky.

[Su1] T. Suffridge, The principle of subordination applied to functions of several variables, Pacific J. Math. **33** (1970), 241-248.

[Su2] _______, Starlikeness, convexity and other geometric properties of holomorphic maps in higher dimensions. Complex Analysis (Proc. Conf. Univ. Kentucky, Lexington, Ky 1976), pp. 146-159. Lecture Notes in Math. Vol. 599, Springer Verlag, Berlin-Heidelberg-New York 1977.

[Sz] G. Szegö, Über eine Extremalaufgabe aus der Theorie der schlichten Abbildungen, Sitzungsberichte der Berliner Mathematische Gesellschaft **22** (1923), 38-47. [Gabor Szegö: Collected Papers, ed. by Richard Askey, Birkhäuser Verlag, Boston-Basel-

Stuttgart 1982, Vol. 1, pp. 607-618.]

[Z] M. Zhang, Ein Überdeckungssatz für Konvexe Gebiete, Acad. Sinica Science Record **5** (1952), 17-21.

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