Toeplitz Operators and Carleson Measures on Generalized Bargmann-Fock Spaces

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1. Introduction

1.1. Definitions

Throughout this paper, λ denotes the Lebesgue measure on $\mathbb{C}^n$ and

$$\omega_o = dd^c|z|^2$$

the Euclidean Kähler form in $\mathbb{C}^n$, where $d^c = \frac{\sqrt{-1}}{4}(\bar\partial - \partial)$. Let $\varphi \in \mathscr{C}^2(\mathbb{C}^n)$ be a function, μ a measure in $\mathbb{C}^n$, and $p \in [1, \infty)$. One can define the spaces

$$L^p(e^{-p\varphi}d\mu) \qquad \text{and} \qquad \mathscr{F}^p(\mu, \varphi) := L^p(e^{-p\varphi}d\mu) \cap \mathcal{O}(\mathbb{C}^n).$$

If the measure μ is Lebesgue measure, we simply write

$$\mathscr{F}^p(\lambda, \varphi) =: \mathscr{F}^p(\varphi).$$

Similarly one can define

$$L^\infty(e^{-\varphi}, \mu) = \{f\ ;\ \mu\text{-Ess. Sup.}|f|e^{-\varphi} < +\infty\}$$

and

$$\mathscr{F}^\infty(\varphi) := L^\infty(e^{-\varphi}, \mu) \cap \mathcal{O}(\mathbb{C}^n).$$

When the measure μ has reasonable properties $\mathscr{F}^p(\mu, \varphi) \subset L^p(e^{-p\varphi}d\mu)$ (resp. $\mathscr{F}^\infty(\varphi) \subset L^\infty(e^{-\varphi})$) is a closed subspace. In particular, we have for such μ an orthogonal projection $L^2(e^{-2\varphi}d\mu) \to \mathscr{F}^2(\mu, \varphi)$, which is called the *Bergman projection*. This projection is an integral operator given by an integral kernel called the *Bergman kernel*, here denoted $K(z, \bar w)$. Later we recall basic and well-known properties of the Bergman kernel.

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The following class of operators appears in a number of areas of complex analysis and geometry, as well as in microlocal analysis and in mathematical physics.

DEFINITION 1.1 **(Toeplitz Operator)**. Let K be the Bergman kernel, i.e., the kernel for the orthogonal projection $L^2(e^{-2\varphi}d\lambda) \to \mathscr{F}^2(\varphi)$. For a positive measure μ in $\mathbb{C}^n$, the operator T_μ defined by

$$T_\mu f(z) := \int_{\mathbb{C}^n} K(z, \bar{w}) f(w) e^{-2\varphi(w)} d\mu(w),$$

is called a *Toeplitz operator with symbol* μ. ◇

REMARK 1.2. In the present paper, we apply Toeplitz operators only to holomorphic functions. In other areas of mathematics, such as microlocal analysis and mathematical physics, the name *Toeplitz Operator* refers to the operator $T_\mu \circ P$, where $P : L^2(e^{-\varphi}d\lambda) \to \mathscr{F}^2(\varphi)$ is the Bergman projection. While there are probably good reasons to consider the latter definition, namely that these definitions have good behavior with respect to composition, such precision will not be important in the present paper. ◇

At this stage, we don't want to say too much about the precise meaning of this operator. When we impose conditions on the measure μ, we will be more precise.

DEFINITION 1.3 **(Carleson Measure)**. Let μ be a positive measure on $\mathbb{C}^n$ and fix $p \in [1, \infty)$. We say μ is *Carleson for* $\mathscr{F}^p(\varphi)$ if there exists a positive constant C such that

$$\int_{\mathbb{C}^n} |f|^p e^{-p\varphi} d\mu \le C \int_{\mathbb{C}^n} |f|^p e^{-p\varphi} d\lambda$$

for all $f \in \mathscr{F}^p(\varphi)$. ◇

Thus, by definition, μ is Carleson for $\mathscr{F}^p(\varphi)$ when the inclusion $\iota_\mu : \mathscr{F}^p(\varphi) \hookrightarrow L^p(e^{-p\varphi}d\mu)$ is bounded.

DEFINITION 1.4 **(Vanishing Carleson Measure)**. A measure μ is said to be *vanishing Carleson* if the inclusion $\iota_\mu : \mathscr{F}^p(\varphi) \hookrightarrow L^p(e^{-p\varphi}d\mu)$ is a compact operator. ◇

Observe that vanishing Carleson measures are Carleson.

The goal of the present paper is to study basic properties of the Toeplitz operator T_μ in terms of geometric and/or operator-theoretic properties of the measure μ.

REMARK 1.5. The above definitions are easily extended to the setting of sections of holomorphic line bundles on complex manifolds, but such definitions are too general for the main results of this paper. Indeed, the main results of the paper rely on rather strong decay of the Bergman kernel far from the diagonal. Such decay is not true in the more general setting described above, and it is still not understood whether any kind of general decay result could be formulated. We therefore content ourselves with the setting of these generalized Bargmann-Fock spaces, hoping to return to the more general problem in a future publication. $\diamond$

1.2. Results

The following are the main results of this paper.

THEOREM 1. *Let $\varphi \in \mathscr{C}^2(\mathbb{C}^n)$ and assume that $m\omega_o \le dd^c\varphi \le M\omega_o$ for some positive constants $m < M$. Fix $p \in [1, \infty)$. Then the Toeplitz operator $T_\mu : \mathscr{F}^p(\varphi) \to \mathscr{F}^p(\varphi)$ with symbol μ is everywhere-defined and bounded if and only if μ is a Carleson measure.*

THEOREM 2. *Let $\varphi \in \mathscr{C}^2(\mathbb{C}^n)$ satisfy $m\omega_o \le dd^c\varphi \le M\omega_o$ for some positive constants $m < M$. Fix $p \in [1, \infty)$. Then a Toeplitz operator $T_\mu : \mathscr{F}^p(\varphi) \to \mathscr{F}^p(\varphi)$ with symbol μ is compact if and only if μ is vanishing Carleson.*

REMARK 1.6. The condition $0 < m\omega_o \le dd^c\varphi \le M\omega_o$ will often be denoted simply $dd^c\varphi \simeq \omega_o$. We emphasize that this notation implies *positive* upper and lower bounds. At times we will also consider negative lower bounds, but we will not use the notation $\simeq$ in that situation. $\diamond$

An ingredient in the proofs of these two theorems is a geometric characterization of the Carleson and compact Carleson conditions for a measure. The geometric characterization of Carleson measures in the Hardy space is classical, and in the unweighted Bergman space of the disk and the ball it has been treated more recently by Luecking [Lu-1983] and Duren-Weir [DW-2007] respectively. Most recently we learned from M. Abate that he and Saracco [AS-2009] have treated the case of bounded strictly pseudoconvex domains. The weighted Bergman space of entire functions was treated by Ortega-Cerdà in [O-1998], where he established Theorem 5.1 of Section 5, which states that if $dd^c\varphi \simeq \omega_o$ then a measure μ is Carleson for $\mathscr{F}^p(\varphi)$ if and only if $\mu(D(z,1))$ is bounded above by a constant independent of z. In the same section we show (Theorem 5.2) that a measure μ is vanishing Carleson if and only if $\mu(B(z,1)) \to 0$ as $|z| \to \infty$.

In view of the geometric characterizations of the Carleson conditions, it makes sense to inquire whether T_μ is bounded on $\mathscr{F}^\infty(\varphi)$ if and only if μ is Carleson. This is indeed true, and similar remarks hold for the vanishing

Carleson case. Therefore Theorems 1 and 2 hold for $p = \infty$ when the Carleson condition is the geometric one.

In Theorems 1 and 2 as well as at other points of the paper, a key tool we use is an off-diagonal exponential decay property for the Bergman kernel established by M. Christ [C-1991] in the case $n = 1$ and by H. Delin [D-1998] more generally.

The approach to handling weights whose curvature is uniformly comparable to the Euclidean metric form was initiated by Berndtsson and Ortega-Cerdà in [BO-1995], and a number of the techniques we used here were inspired by this approach. Though the paper [BO-1995] takes place in dimension 1, some of the results were extended by Lindholm [L-2001] to higher dimensions, and others are easy to modify, as we show here.

In Section 6 we recall the definition of the so-called *Berezin transform* of a measure μ. We will then prove two results that characterize Carleson measures and vanishing Carleson measures respectively in terms of growth properties of their Berezin transforms. Interestingly, the exponential decay of the Bergman kernel far from the diagonal does not factor into these theorems, which parses well with the fact that the analogous results were obtained by Abate and Saracco for strongly pseudoconvex domains in $\mathbb{C}^n$.

In the case $n = 1$ and $\varphi(z) = |z|^2$, our results are all contained in the paper [IZ-2010] of Isralovitz and Zhu, which further explores so-called Schatten class membership for a Toeplitz operator. As mentioned in that paper, there is no major difference between $n = 1$ and $n \geq 2$, but there are in fact rather vast differences when one changes the weight. Indeed, the Bergman kernel for the specialized weight $\varphi(z) = |z|^2$ is known explicitly, and has the property

$$|K(z, \bar{w})| e^{-|z|^2 - |w|^2} = e^{-\frac{1}{2}|z-w|^2}.$$

But for the class of weights we consider, this quadratic decay is known not to hold (even in dimension $n = 1$), and is expected to be very rare [C-1991]. Fortunately, however, we shall see that such rapid decay is not necessary.

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2. Weights with Euclidean curvature bounds

In this section we discuss well-known results about local weighted L^p estimates for weights φ such that $dd^c\varphi$ is uniformly bounded above. Most of the proofs are standard, and at best there are only a few minor innovations on previous methods.

2.1. Solving the dd^c-equation with uniform estimates

The following lemma is fundamental.

LEMMA 2.1. *There exists a constant $C > 0$ with the following property. Let ω be a $\mathscr{C}^2$-smooth, closed $(1,1)$-form on a neighborhood of the closed unit ball $\overline{B}$ in $\mathbb{C}^n$, such that*

$$-M\omega_o \le \omega \le M\omega_o$$

for some positive constant M. Then there exist a function $\psi \in \mathscr{C}^2(B)$ such that

$$dd^c\psi = \omega \quad \textit{and} \quad \sup_B(|\psi| + |d\psi|) \le CM.$$

Proof. We assume that ω has compact support in $B(0,2)$. Suppose first that $n = 1$. Then one simply takes

$$\psi(z) := \frac{1}{\pi}\int_{B(0,2)} \log|z-\zeta|^2\omega(\zeta).$$

Note that $\omega = h\omega_o$ for some real-valued function h. A standard argument using integration-by-parts shows that

$$\frac{\partial^2\psi}{\partial z\partial\bar{z}} = h.$$

The function ψ is clearly bounded by the constant

$$M \sup_{z\in B(0,1)} \int_{B(0,2)} \left|\log|\zeta - z|^2\right| dA$$

while the derivative is controlled by

$$M \sup_{z\in B(0,1)} \int_{B(0,2)} \frac{dA(\zeta)}{|z-\zeta|}.$$

Thus we have the stated result.

In higher dimensions, write $\omega = \sum_{i,j} \omega_{i\bar{j}} \frac{\sqrt{-1}}{2} dz^i \wedge d\bar{z}^j$. Then as in the 1-dimensional case, the function

$$\psi(z) := \frac{1}{\pi}\int_{B(0,2)} \omega_{1\bar{1}}(\zeta, z^2, ..., z^n)\log|z^1-\zeta|^2 dA(\zeta)$$

then satisfies

$$\frac{\partial^2\psi}{\partial z^1\partial\bar{z}^1} = \omega_{1\bar{1}}.$$

From the condition $d\omega = 0$, we see that, when either i or j (or both) are different from 1,

$$\begin{aligned}\frac{\partial^2\psi}{\partial z^i\partial \bar{z}^j} &= \frac{1}{\pi}\int_{B(0,2)} \frac{\partial^2\omega_{1\bar{1}}}{\partial z^i\partial \bar{z}^j}(\zeta, z^2, ..., z^n)\log|z^1-\zeta|^2 dA(\zeta)\\ &= \frac{1}{\pi}\int_{B(0,2)} \frac{\partial^2\omega_{i\bar{j}}}{\partial \zeta\partial \bar{\zeta}}(\zeta, z^2, ..., z^n)\log|z^1-\zeta|^2 dA(\zeta)\\ &= \omega_{i\bar{j}}(z).\end{aligned}$$

As before, ψ is bounded in $\mathscr{C}^1$-norm by CM. The proof is complete. □

2.2. Uniform local pluriharmonic recentering of weights

COROLLARY 2.2. *Let $z \in \mathbb{C}^n$ and let $\varphi \in \mathscr{C}^2(B(z,2))$ satisfy $-M\omega_o \le dd^c\varphi \le M\omega_o$ for some positive constant M. Then there exists a function $F \in \mathcal{O}(B(z,3/2))$ and a constant $C > 0$ that depends on M but not on z, such that $F(z) = 0$ and*

$$\sup_{B(z,1)} \left(|\varphi - \varphi(z) - \text{Re } F| + |\nabla(\varphi - \varphi(z) - \text{Re } F)|\right) \le C.$$

Proof. By translation we may assume $z = 0$. Apply Lemma 2.1 to the form $\omega = dd^c\varphi$ to obtain a function ψ such that $dd^c\psi = dd^c\varphi$ with the appropriate $\mathscr{C}^1$-estimates. The function $\eta := \varphi - \varphi(0) + \psi(0) - \psi$ is then pluriharmonic, and therefore is the real part of a holomorphic function F. The imaginary part of F can be taken to be the function $\int_0^z d^c\eta$, and so vanishes at 0. Then

$$|\varphi - \varphi(z) - \text{Re } F| = |\psi - \psi(0)| \le C$$

for some constant $C > 0$ depending only on M. □

2.3. Weighted Bergman inequalities

PROPOSITION 2.3 **(Weighted Bergman inequalities)**. *Let $\varphi \in \mathscr{C}^2(\mathbb{C}^n)$ satisfy $-M\omega_o \le dd^c\varphi \le M\omega_o$ for some positive constant M. Then for each $r > 0$ there exists a constant C_r such that if $f \in \mathscr{F}^p(\varphi)$ then*

$$(|f|^p e^{-p\varphi})(z) \le C_r^p \int_{B(z,r)} |f|^p e^{-p\varphi} d\lambda \tag{1}$$

and

$$|\nabla(|f|^p e^{-p\varphi})|(z) \le C_r^p \int_{B(z,r)} |f|^p e^{-p\varphi} d\lambda \tag{2}$$

Proof. Using Corollary 2.2, we may write

$$|f|e^{-\varphi} = |fe^{-F}|e^{-\varphi(z)}e^{-\varphi+\varphi(z)+\text{Re } F}.$$

Since $e^{-\varphi+\varphi(z)+\text{Re } F}$ is bounded in $\mathscr{C}^1$-norm, the proof reduces to the unweighted case, in which the result is classical. □

2.4. Slow growth of Bergman functions

COROLLARY 2.4. *Let φ be as in Proposition 2.3, and let $a > 0$. Then there exists $\varepsilon > 0$ with the following property. If $z \in \mathbb{C}^n$, $f \in \mathscr{F}^p(\varphi)$, $||f||_p \leq 1$ and $|f(z)|e^{-\varphi(z)} \geq a$ then $|f(w)|e^{-\varphi(w)} \geq a/2$ for all $w \in D(z, \varepsilon)$.*

Proof. Otherwise (2) in Proposition 2.3 is violated. □

2.5. One-point interpolation with uniform L^p estimates

PROPOSITION 2.5 **(Uniform 1-point interpolation in $\mathscr{F}^p(\varphi)$).** *Suppose that $\varphi \in \mathscr{C}^2(\mathbb{C}^n)$ satisfies $dd^c\varphi \simeq \omega_o$. Then there exists $C > 0$ such that for each $z \in \mathbb{C}^n$ there exists $f \in \mathscr{F}^p(\varphi)$ such that $f(z) = e^{\varphi(z)}$ and $||f||_p \leq C$.*

Proof. Let $z \in \mathbb{C}^n$, and fix a smooth function $\chi \in \mathscr{C}^\infty_o(B(z,2))$ taking values in $[0,1]$, such that $\chi|_{B(z,1)} \equiv 1$ and $|\bar\partial\chi| \leq 3$. Consider the $(0,1)$-form $\theta(\zeta) = e^{F(\zeta)+\varphi(z)}\bar\partial\chi(\zeta)$, where $F \in \mathcal{O}(B(z,2))$ vanishes at z and satisfies the estimate $|\varphi - \varphi(z) - \text{Re } F| \leq c$ for some positive constant c independent of z. Such an F exists by Corollary 2.2.

Next fix $\varepsilon > 0$ such that $dd^c(\varphi - \varepsilon|\cdot - z|^2) \simeq \omega_o$. Since θ is supported on a z-centered spherical annulus,

$$\int_{\mathbb{C}^n} \frac{|\theta|^2 e^{-2(\varphi-\varepsilon|\cdot-z|^2)}}{|\cdot - z|^{2n}} d\lambda \leq 9^n e^{4\varepsilon} \int_{1\leq|\zeta-z|\leq 2} e^{2(\text{Re } F+\varphi(z)-\varphi)} d\lambda \leq C_o$$

for some $C_o > 0$. By Hörmander's Theorem applied to the weight

$$\varphi - \varepsilon|\cdot - z|^2 + \log|\cdot - z|^n,$$

whose curvature is uniformly strictly positive (with respect to z as well) there exists u such that $\bar\partial u = \theta$ and

$$\int_{\mathbb{C}^n} \frac{|u|^2 e^{-2(\varphi(\zeta)-\varepsilon|\zeta-z|^2)}}{|\zeta - z|^{2n}} d\lambda(\zeta) \leq C' \qquad (3)$$

for some constant C' independent of z. In particular, $u(z) = 0$.

Let

$$f(\zeta) := e^{\varphi(z)+F(\zeta)}\chi(\zeta) - u(\zeta).$$

Then $f(z)e^{-\varphi(z)} = 1$, $\bar\partial f = 0$, and

$$\int_{\mathbb{C}^n} |f(\zeta)|^2 e^{-(2\varphi(\zeta)-\varepsilon|\zeta-z|^2)} d\lambda(\zeta)$$
$$\lesssim \int_{\mathbb{C}^n} |\chi(\zeta)|^2 e^{\varepsilon|\zeta-z|^2-2(\varphi(\zeta)-\varphi(z)-\text{Re } F(\zeta))} d\lambda(\zeta)$$
$$+ \int_{\mathbb{C}^n} \frac{|u(\zeta)|^2 e^{-2(\varphi(\zeta)-\varepsilon|\zeta-z|^2)}}{|\zeta - z|^{2n}} |z - \zeta|^{2n} e^{-\varepsilon|\zeta-z|^2} d\lambda(\zeta).$$

The first of these integrals is uniformly bounded because $e^{\varepsilon|\zeta - z|^2}$ and $|\varphi(\zeta) - \varphi(z) - \operatorname{Re} F|$ are bounded uniformly in z on the support of χ, while the second integral is uniformly bounded because of (3) and the bound $r^{2n}e^{-\varepsilon r} \le (2n/\varepsilon)^{2n}e^{-2n}$.

Now, by (1) of Proposition 2.3 we have the estimate

$$\begin{aligned} |f(\zeta)|^2 e^{-2\varphi(\zeta)} &\le |f(\zeta)|^2 e^{-(2\varphi(\zeta)-\varepsilon|\zeta-z|^2)} d\lambda \\ &\le C \int_{B(\zeta,1)} |f|^2 e^{-(2\varphi-\varepsilon|\cdot - z|^2)} d\lambda \le \tilde{C}, \end{aligned}$$

so that $||f||_{\infty,\varphi} \le C$ for some uniform constant C. This establishes the case $p = \infty$.

On the other hand, we also have that for any $w \in \mathbb{C}^n$

$$|f(w)|^2 e^{-2(\varphi(w)-\varepsilon|w-z|^2)} \lesssim \int_{B(w,1)} |f|^2 e^{-2(\varphi-\varepsilon|\cdot - z|^2)} d\lambda \le C^2$$

for some constant C independent of w. Therefore

$$\int_{\mathbb{C}^n} |f|^p e^{-p\varphi} d\lambda = \int_{\mathbb{C}^n} \left(|f|^2 e^{-(2\varphi-\varepsilon|\cdot - z|^2)}\right)^{p/2} e^{-\frac{p\varepsilon}{2}|\cdot - z|^2} d\lambda \le C^p C_1,$$

and thus $||f||_{p,\varphi}$ is bounded uniformly in z, as claimed. □

REMARK 2.6. Note that the constant in Proposition 2.5 bounding $||f||_{p,\varphi}$ is also independent of $p \in [1, \infty)$. Although we will not need this independence, it is an example of one of the themes of this paper, namely p-independence of many conditions; for instance, we will see that the Carleson condition, and therefore the boundedness of a Toeplitz operator, does not depend on p. ◇

3. The Bergman kernel

Let $\varphi : \mathbb{C}^n \to \mathbb{R}$ be a $\mathscr{C}^2$-smooth weight function such that $dd^c\varphi \simeq \omega_o$ (recall Remark 1.6). Then the subspace $\mathscr{F}^2(\varphi)$ of $L^2(e^{-2\varphi}d\lambda)$ is closed. The orthogonal projection $P : L^2(e^{-2\varphi}d\lambda) \to \mathscr{F}^2(\varphi)$, called the Bergman projection, is therefore bounded. The projection P is given by integration against a kernel $K(z, \bar{w})$, and by usual Hilbert space theory

$$K(z, \bar{w}) = \sum_{j=1}^{\infty} f_j(z)\overline{f_j(w)},$$

where $\{f_j\}$ is any orthonormal basis for $\mathscr{F}^2(\varphi)$. If we fix $z \in \mathbb{C}^n$ then we can select a basis $\{g_j\}_{j\ge 2}$ for the subspace $S_z \subset \mathscr{F}^2(\varphi)$ of all weighted-squared-integrable holomorphic functions vanishing at z. The sub-mean value property for holomorphic functions (or (1) of Proposition 2.3 for $p = 2$) shows that evaluation at z is a bounded linear functional (where on

$\mathbb{C}$ we put the norm $|\cdot|e^{-\varphi(z)}$) and therefore S_z has codimension 1 or 0. Since $dd^c\varphi \geq c\omega_o$, there are non-vanishing holomorphic functions at any point z, and therefore $\mathscr{F}^2(\varphi) = S_z \oplus \mathbb{C}f_1$ for some $f_1 \in \mathscr{F}^2(\varphi)$ with $||f_1||_{2,\varphi} = 1$, unique up to a unimodular constant. We therefore have

$$K(z,\bar{w}) = f_1(z)\overline{f_1(w)} + \sum_{j\geq 2} g_j(z)\overline{g_j(w)} = f_1(z)\overline{f_1(w)}.$$

We note in particular that

$$K(z,\bar{z}) = \sup_{||f||=1} |f(z)|^2,$$

and that the supremum is actually a maximum. We therefore obtain from (1) of Proposition 2.3 the following proposition.

PROPOSITION 3.1. *There is a constant $C > 0$ such that*

$$K(z,\bar{z})e^{-2\varphi(z)} \leq C$$

for all $z \in \mathbb{C}^n$. Therefore

$$|K(z,\bar{w})|e^{-\varphi(z)-\varphi(w)} \leq C$$

for all $z, w \in \mathbb{C}^n$.

3.1. Off-dagonal estimates for the Bargmann-Fock Bergman kernel

Below we will make extensive use of the following result, due to M. Christ [C-1991] in the case $n = 1$ and to H. Delin [D-1998] for $n \geq 2$.

THEOREM 3.2 **(Christ, Delin)**. *Let $\varphi \in \mathscr{C}^2(\mathbb{C}^n)$ satisfy $dd^c\varphi \simeq \omega_o$ and let K denote the Bergman kernel, i.e., the integral kernel for the orthogonal projection $L^2(e^{-2\varphi}d\lambda) \to \mathscr{F}^2(\varphi)$. Then there are constants $\varepsilon, C > 0$ such that for all $z, w \in \mathbb{C}^n$,*

$$|K(z,\bar{w})|e^{-\varphi(z)-\varphi(w)} \leq Ce^{-\varepsilon|z-w|}.$$

Since it is so central to our paper, we shall give a proof of Theorem 3.2 in Appendix A. Our proof is similar in spirit to Delin's but is rather more streamlined. It still makes use of Berndtsson's key idea about twisted estimates for the L^2-minimal solution of the $\bar{\partial}$ equation, which we will also review in the same appendix for the sake of convenient access.

While Theorem 3.2 gives upper bounds for the Bergman kernel far from the diagonal, the following easier result gives lower bounds in a small but uniform neighborhood of the diagonal.

PROPOSITION 3.3. *Let $\varphi \in \mathscr{C}^2(\mathbb{C}^n)$ satisfy $dd^c\varphi \simeq \omega_o$ and denote by $K(z,\bar{w})$ the kernel of the Bergman projection $P : L^2(e^{-2\varphi}d\lambda) \to \mathscr{F}^2(\varphi)$. Then there exist positive constants ε, C_1 and C_2 such that for each $z \in \mathbb{C}^n$ and each $w \in B(z,\varepsilon)$,*

$$|K(z,\bar{w})|e^{-(\varphi(z)+\varphi(w))} \geq C_1|K(z,\bar{z})|e^{-2\varphi(z)} \geq C_2.$$

Proof. As $K(z,\bar{z})e^{-2\varphi(z)} = \sup_{||f||_2=1} |f(z)|^2 e^{-2\varphi(z)}$, Proposition 2.5 shows that $K(z,\bar{z})e^{-2\varphi(z)} \geq C_o$ for some $C_o > 0$ independent of z. Fixing z now, consider the function $F(w) = K(w,\bar{z})e^{-\varphi(z)}$. Then $F \in \mathscr{F}^2(\varphi)$, $||F||_2^2 = K(z,\bar{z})e^{-2\varphi(z)} \lesssim 1$ by Proposition 3.1 and the reproducing property of the Bergman kernel, and $|F(z)|e^{-\varphi(z)} \geq C_o$. By Corollary 2.4, there exists $C, \varepsilon > 0$ independent of z such that

$$|F(w)|e^{-\varphi(w)} \geq C|F(z)|e^{-\varphi(z)}$$

for all $w \in B(z,\varepsilon)$. The proof is finished. □

3.2. Boundedness of the Bergman projection on $\mathscr{F}^p(\varphi)$

An easy consequence of the off-diagonal decay of the Bergman kernel is the following result.

PROPOSITION 3.4. *Let $p \in [1,\infty]$. Then the Bergman projection is bounded as a map from $L^p(e^{-p\varphi}d\lambda)$ to $\mathscr{F}^p(\varphi)$.*

Proof. If $p = \infty$ then for F such that $|F|e^{-\varphi} \in L^\infty(\mathbb{C}^n)$ we have

$$\begin{aligned} |PF(z)|e^{-\varphi} &= \left| \int_{\mathbb{C}^n} F(w)K(z,\bar{w})e^{-2\varphi(w)}e^{-\varphi(z)}d\lambda \right| \\ &\leq ||F||_{\infty,\varphi} \int_{\mathbb{C}^n} |K(z,\bar{w})|e^{-(\varphi(z)+\varphi(w))}d\lambda(w) \\ &\lesssim ||F||_{\infty,\varphi} \int_{\mathbb{C}^n} e^{-\varepsilon|z-w|}d\lambda(w) \lesssim ||F||_{\infty,\varphi}, \end{aligned}$$

and thus $||PF||_{\infty,\varphi} \lesssim ||F||_{\infty,\varphi}$. If $p \in [1,\infty)$ then for $F \in L^p(e^{-p\varphi}d\lambda)$ we have

$$\begin{aligned} &\int_{\mathbb{C}^n} \left| \int_{\mathbb{C}^n} F(w)K(z,\bar{w})e^{-2\varphi(w)}d\lambda(w) \right|^p e^{-p\varphi(z)}d\lambda(z) \\ &\leq \int_{\mathbb{C}^n} \left(\int_{\mathbb{C}^n} |F(w)|e^{-\varphi(w)}|K(z,\bar{w})|e^{-(\varphi(z)+\varphi(w))}d\lambda(w) \right)^p d\lambda(z) \\ &\lesssim \int_{\mathbb{C}^n} \left(\int_{\mathbb{C}^n} |F(w)|e^{-\varphi(w)}e^{-\varepsilon|z-w|}d\lambda(w) \right)^p d\lambda(z) \\ &\leq \int_{\mathbb{C}^n} \left(\left(\int_{\mathbb{C}^n} e^{-\varepsilon|z-\cdot|}d\lambda \right)^{p-1} \int_{\mathbb{C}^n} |F|^p e^{-p\varphi}e^{-\varepsilon|z-\cdot|}d\lambda \right) d\lambda(z) \\ &\lesssim \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} |F(w)|^p e^{-p\varphi(w)}e^{-\varepsilon|z-w|}d\lambda(w)d\lambda(z) \\ &\lesssim \int_{\mathbb{C}^n} |F(w)|^p e^{-p\varphi(w)}d\lambda(w) \end{aligned}$$

This is what was claimed. □

3.3. Dense subspaces of Bergman space

Let

$$\mathscr{F}_0^\infty(\varphi) := \left\{ f \in \mathscr{F}^\infty(\varphi) \; ; \; \lim_{|z|\to\infty} |f(z)|e^{-\varphi(z)} \to 0 \right\}.$$

We begin with the following proposition established by Lindholm in [L-2001].

PROPOSITION 3.5. *Fix $\varphi \in \mathscr{C}^2(\mathbb{C}^n)$ satisfying $dd^c\varphi \simeq \omega_o$. Then the following hold.*

1. *For each $f \in \mathscr{F}^\infty(\varphi)$ there is a sequence $f_j \subset \mathscr{F}^2(\varphi)$ such that $f_j \to f$ locally uniformly.*
2. *$\mathscr{F}_0^\infty(\varphi) \cap \mathscr{F}^2(\varphi)$ is dense in $\mathscr{F}_0^\infty(\varphi)$.*

Proof. Fix $f \in \mathscr{F}^\infty(\varphi)$. We are going to find $\{f_j \; ; j = 1, 2, ...\} \subset \mathscr{F}^2(\varphi)$ such that

$$\sup_{\mathbb{C}^n} |f_j - f|e^{-\varphi} \to 0 \text{ as } j \to \infty.$$

Let $\chi \in \mathscr{C}_0^\infty(B(0,1))$ satisfy $0 \le \chi \le 1$ as well as $\chi|_{B(0,1/2)} \equiv 1$, and set $\chi_j(z) = \chi(z/j)$. Let

$$f_j := P(\chi_j f) \qquad \text{and} \qquad u_j := \chi_j f - P(\chi_j f).$$

Then $f_j \in \mathscr{F}^2(\varphi)$, and u_j is the unique solution of the equation $\bar\partial u = \bar\partial \chi_j f$ whose $L^2(e^{-2\varphi} d\lambda)$-norm is minimal. By Berndtsson's Theorem (which is stated as Theorem A.5 in the appendix) one has the estimate

$$\sup_{\mathbb{C}^n} |u_j|e^{-\varphi} \lesssim \sup_{\mathbb{C}^n} (||\bar\partial\chi_j|| \cdot |f|e^{-\varphi}) \lesssim \frac{1}{j} \sup_{\mathbb{C}^n} |f|e^{-\varphi} \to 0.$$

It follows that for each $z \in \mathbb{C}^n$,

$$\begin{aligned} |f(z) - f_j(z)|e^{-\varphi(z)} &\le |f(z) - \chi_j(z)f(z)|e^{-\varphi(z)} + \sup_{\mathbb{C}^n} |u_j|e^{-\varphi} \\ &\le \left|(1 - \chi(\tfrac{z}{j}))f(z)\right| e^{-\varphi(z)} + O(j^{-1}), \end{aligned}$$

from which both conclusions are clearly deduced. □

Since $\mathscr{F}^p(\varphi) \subset \mathscr{F}_0^\infty(\varphi) \subset \mathscr{F}^\infty(\varphi)$, the first containment being a consequence of (1) in Proposition 2.3, one has the following corollary.

COROLLARY 3.6. *Fix $p \in [1, \infty)$ and $\varphi \in \mathscr{C}^2(\mathbb{C}^n)$ satisfying $dd^c\varphi \simeq \omega_o$. Then $\mathscr{F}^p(\varphi) \cap \mathscr{F}^2(\varphi)$ is dense in $\mathscr{F}^p(\varphi)$ and $\mathscr{F}_0^\infty(\varphi) \cap \mathscr{F}^2(\varphi)$ is dense in $\mathscr{F}_0^\infty(\varphi)$.*

COROLLARY 3.7. *Suppose $\varphi \in \mathscr{C}^2(\mathbb{C}^n)$ and $dd^c\varphi \simeq \omega_o$. Then for every $p \in [1, \infty]$, $P|_{\mathscr{F}^p(\varphi)} = \mathrm{id}$.*

Proof. First suppose $p \in [1, \infty)$. Let $f \in \mathscr{F}^p(\varphi)$ and let $f_n \in \mathscr{F}^2(\varphi)$ such that $f_n \to f$ in $\mathscr{F}^p(\varphi)$. Since P is bounded on $\mathscr{F}^p(\varphi)$, $Pf = P(\lim f_n) = \lim P f_n = \lim f_n = f$.

Next let $f \in \mathscr{F}^\infty$. Consider the sequence $f_j \subset \mathscr{F}^2(\varphi)$ constructed in the proof of Proposition 3.5, which has the properties that $f_j \to f$ locally uniformly and also $|f - f_j|e^{-\varphi} \le ||f||_\infty + O(1/j)$. Then for each $z \in \mathbb{C}^n$,

$$\begin{aligned} &|P(f - f_n)(z)|e^{-\varphi(z)} \\ &= \left| \int_{\mathbb{C}^n} (f(w) - f_n(w))K(z, \bar{w})e^{-2\varphi(w)-\varphi(z)} d\lambda(w) \right| \\ &\le \int_{\mathbb{C}^n} |f(w) - f_n(w)|e^{-\varphi(w)}e^{-\varepsilon|z-w|} d\lambda(w) \\ &= \int_{|w|\le j} |f(w) - f_n(w)|e^{-\varphi(w)}e^{-\varepsilon|z-w|} d\lambda(w) \\ &\qquad + \int_{|w|>j} |f(w) - f_n(w)|e^{-\varphi(w)}e^{-\varepsilon|z-w|} d\lambda(w) \end{aligned}$$

which tends to 0 as $n \to \infty$, locally uniformly. Thus, since $f_n \in \mathscr{F}^2(\varphi)$ and is thus reproduced by P, we have

$$|Pf(z) - f(z)|e^{-\varphi(z)} \le |P(f - f_n)(z)|e^{-\varphi(z)} + |f_n(z) - f(z)|e^{-\varphi(z)}.$$

The right hand side converges to 0 as $n \to \infty$, so $Pf = f$, as claimed. □

3.4. The dual space of $\mathscr{F}^p(\varphi)$

Let $p \in [1, \infty)$ and let $q = p/(p-1)$ be its dual exponent. Given a function $g \in \mathscr{F}^q(\varphi)$, Hölder's Inequality implies that the linear functional $\Psi_g : \mathscr{F}^p(\varphi) \to \mathbb{C}$ given by

$$\Psi_g(f) := \int_{\mathbb{C}^n} f\bar{g}e^{-2\varphi} d\lambda \tag{4}$$

is bounded. Let $\Psi : \mathscr{F}^q(\varphi) \to \mathscr{F}^p(\varphi)^*$ be the map sending g to Ψ_g.

PROPOSITION 3.8. *Let $p \in [1, \infty)$ and let $q = p/(p-1)$ be the dual exponent. The map $\Psi : \mathscr{F}^q(\varphi) \to \mathscr{F}^p(\varphi)^*$ is bijective, and moreover there is a constant C such that $C||g||_q \le ||\Psi_g|| \le ||g||_q$ holds for all $g \in \mathscr{F}^q(\varphi)$.*

Proof. Let us begin with surjectivity. Let $\ell \in \mathscr{F}^p(\varphi)^*$. By Hahn-Banach, we can extend ℓ to an element $\tilde{\ell} \in L^p(e^{-p\varphi}d\lambda)^*$ such that $||\tilde{\ell}|| = ||\ell||$. By the Riesz Representation Theorem there exists $G \in L^q(e^{-q\varphi}d\lambda)$ such that

$$\ell(f) = \int_{\mathbb{C}^n} f\bar{G}e^{-2\varphi} d\lambda, \quad f \in \mathscr{F}^p(\varphi).$$

Let $g := PG$. Then for $f \in \mathscr{F}^p(\varphi)$,

$$\begin{aligned}
&\int_{\mathbb{C}^n} f\bar{g}e^{-2\varphi}d\lambda \\
&= \int_{\mathbb{C}^n}\int_{\mathbb{C}^n} f(w)K(z,\bar{w})\overline{G(z)}e^{-2\varphi(z)}e^{-2\varphi(w)}d\lambda(z)d\lambda(w) \\
&= \int_{\mathbb{C}^n} f(z)\overline{G(z)}e^{-2\varphi(z)}d\lambda(z),
\end{aligned}$$

where the second equality follows from Corollary 3.7. We conclude that $\ell = \Psi_g$, i.e., Ψ is surjective.

The upper bound $||\Psi_g|| \le ||g||_q$ is an immediate consequence of Hölder's Inequality. Of course, the lower bound in the claimed estimate implies the desired injectivity, so we are done when we establish this lower bound.

To get the lower bound, fix $g \in \mathscr{F}^q(\varphi) \subset L^q(e^{-q\varphi}d\lambda)$. Then Ψ_g extends to a linear functional $\tilde{\Psi}_g : L^p(e^{-p\varphi}d\lambda) \to \mathbb{C}$ given by the same formula (4). By Alaoglu's Theorem there exists $F_g \in L^p(\varphi)$ such that

$$||F_g||_{p,\varphi} = 1 \qquad \text{and} \qquad \int_{\mathbb{C}^n} F_g\bar{g}e^{-2\varphi}d\lambda = ||g||_{q,\varphi}$$

Therefore

$$||\Psi_g|| \ge \frac{|\Psi_g(PF_g)|}{||PF_g||_{p,\varphi}} = \frac{||g||_{q,\varphi}}{||PF_g||_{p,\varphi}}.$$

But by Proposition 3.4 there exists $C > 0$ such that

$$||PF_g||_{p,\varphi} \le C^{-1}||F_g|| = C^{-1},$$

as desired. □

PROPOSITION 3.9. *$\mathscr{F}_0^\infty(\varphi)^* = \mathscr{F}^1(\varphi)$. Moreover, the map*

$$\Psi : \mathscr{F}^1(\varphi) \mapsto \mathscr{F}_0^\infty(\varphi)^*$$

sending g to Ψ_g defined by (4) *is bijective and satisfies*

$$C||g||_1 \le ||\Psi_g|| \le ||g||_1.$$

Proof. Let $\ell \in \mathscr{F}_0^\infty(\varphi)^*$. We first show that $\ell = \Psi_g$ for some $g \in \mathscr{F}^1(\varphi)$. To this end, by (1) of Proposition 2.3 (for very small r and very large $|z|$) $\mathscr{F}^2(\varphi) \subset \mathscr{F}_0^\infty(\varphi)$, and we know that this subset is dense. Restricting ℓ to $\mathscr{F}^2(\varphi)$, we see from Proposition 3.8 that there exists $g \in \mathscr{F}^2(\varphi)$ such that $\ell(f) = \int f\bar{g}e^{-2\varphi}d\lambda$.

Now let $h \in L^\infty(e^{-\varphi})$ have compact support. Then h is also in $L^2(e^{-2\varphi}d\lambda)$, and since

$$|Ph(z)|e^{-\varphi(z)} \le ||h||_{\infty,\varphi}\int_{\text{Supp}(h)} e^{-\varepsilon|z-w|}d\lambda(w),$$

$Ph \in \mathscr{F}_0^\infty(\varphi)$. We thus have

$$\begin{aligned}
&\left|\int_{\mathbb{C}^n} g(z)\overline{h(z)}e^{-2\varphi(z)}d\lambda(z)\right| \\
&= \left|\int_{\mathbb{C}^n}\int_{\mathbb{C}^n} g(w)K(z,\bar{w})\overline{h(z)}e^{-2\varphi(z)}d\lambda(w)d\lambda(z)\right| \\
&= \left|\int_{\mathbb{C}^n} g(z)\overline{Ph(z)}e^{-2\varphi(z)}d\lambda(z)\right| \\
&= |\ell(Ph)| \lesssim ||Ph||_{\infty,\varphi} \lesssim ||h||_{\infty,\varphi}.
\end{aligned}$$

It follows that in fact $g \in \mathscr{F}^1(\varphi)$. Finally, if $f \in \mathscr{F}_0^\infty(\varphi)$, let $f_n \in \mathscr{F}^2(\varphi)$ be a sequence such that $f_n \to f$ in $\mathscr{F}_0^\infty(\varphi)$. Then

$$|\ell f - \langle f,g\rangle| \le |\ell(f-f_n)| + |\langle f_n - f, g\rangle| \lesssim ||f_n - f||_{\infty,\varphi}.$$

Finally, the upper estimate for $||\Psi||$ is again by Hölder's Inequality. The lower estimate is obtained as follows. Let $g \in \mathscr{F}^1(\varphi)$ and take

$$F := \begin{cases} \frac{g}{|g|e^{-\varphi}} & g \neq 0 \\ 0 & g = 0 \end{cases}$$

Then $|F|e^{-\varphi} \in L^\infty(\mathbb{C}^n)$. Let χ be a smooth cutoff function such that $0 \le \chi \le 1$, $\chi(z) = 1$ when $|z| \le 1/2$ and $\chi(z) = 0$ when $|z| \ge 1$. Define $F_R(z) = \chi(z/R)F(z)$ and $f_R := PF_R$. Since $F_R \in L^2(e^{-2\varphi}d\lambda)$, $f_R \in \mathscr{F}_0^\infty(\varphi)$ and moreover

$$\sup_{\mathbb{C}^n} |f_R|e^{-\varphi} \le C_o \sup_{\mathbb{C}^n} |F_R|e^{-\varphi} = C_o.$$

It follows that, with $\chi_R(z) := \chi(\frac{z}{R})$,

$$\begin{aligned}
||\Psi_g||_\infty &\ge \frac{|\Psi_g(f_R)|}{\sup_{\mathbb{C}^n}|f_R|e^{-\varphi}} \ge \frac{1}{C_o}\int_{\mathbb{C}^n} P(\chi_R F)\bar{g}e^{-2\varphi}d\lambda \\
&= \frac{1}{C_o}\int_{\mathbb{C}^n} \chi_R F\overline{Pg}e^{-2\varphi}d\lambda \ge \frac{1}{C_o}\int_{B(0,R/2)} |g|e^{-\varphi}d\lambda,
\end{aligned}$$

and letting $R \to \infty$ gives the desired estimate. □

4. Compactness in Bergman spaces

In this section we review some of the basic facts about compactness of operators in the setting of Bergman spaces. We begin with the following basic proposition.

PROPOSITION 4.1. *Let $p \in [1,\infty]$ and suppose $dd^c\varphi \simeq \omega_o$. A sequence $\{f_n\} \subset \mathscr{F}^p(\varphi)$ converges weakly to 0 if and only if $n \mapsto ||f_n||_{p,\varphi}$ is bounded and $f_n \to 0$ locally uniformly in $\mathbb{C}$.*

Proof. Let q be the conjugate exponent of p. Suppose first that $f_n \to 0$ weakly in $\mathscr{F}^p(\varphi)$. Let $\ell_n := \Psi_{f_n}$. Then $\{\ell_n\} \subset \mathscr{F}^q(\varphi)^*$ if $p > 1$, and if $p = 1$ then $\{\ell_n\} \subset \mathscr{F}^\infty_0(\varphi)^*$. The sequence, being weakly convergent, is bounded on every element of $\mathscr{F}^q(\varphi)$ if $p > 1$ and $\mathscr{F}^\infty_0(\varphi)$ if $p = 1$, and therefore by the uniform boundedness principle it is bounded. Since Ψ is bounded above and below, we see that $n \mapsto ||f_n||_{p,\varphi}$ is bounded. Therefore, by (1) of Lemma 2.3, $|f_n|$ is locally uniformly bounded, and hence by Montel's Theorem there is a convergent subsequence. Since $f_n \to 0$ weakly, we must have $f_n \to 0$ locally uniformly in $\mathbb{C}^n$.

Conversely suppose $n \mapsto ||f_n||_{p,\varphi}$ is bounded and $f_n \to 0$ locally uniformly in $\mathbb{C}^n$. Let $\{f_{n_k}\}$ be a subsequence of $\{f_n\}$. Since Ψ is bounded above and below, $\ell_{n_k} := \Psi_{f_{n_k}}$ is also bounded. By Alaouglu's Theorem there is a subsequence $f_{n_{k_j}}$ such that $\ell_{n_{k_j}}$ converges to some ℓ weak*. Since Ψ is a bijection, we have $\ell = \Psi_g$ for some $g \in \mathscr{F}^p(\varphi)$. Since $f_n \to 0$ locally uniformly, we must have $g \equiv 0$. Thus every subsequence of $\{f_n\}$ has a subsequence that converges weakly to 0. It follows that $f_n \to 0$ weakly. □

In particular, we have the following proposition.

PROPOSITION 4.2. *Let $p \in [1, \infty]$ and assume that φ satisfies $dd^c\varphi \simeq \omega_o$. Let*

$$F_\gamma(z) := K(z, \bar{\gamma})e^{-\varphi(\gamma)}$$

and

$$H_\gamma(z) := \frac{K(z, \bar{\gamma})}{K(\gamma, \bar{\gamma})e^{-\varphi(\gamma)}} = \frac{F_\gamma(z)}{K(\gamma, \bar{\gamma})e^{-2\varphi(\gamma)}}.$$

Then $F_\gamma \to 0$ and $H_\gamma \to 0$ weakly in $\mathscr{F}^p(\varphi)$ as $\gamma \to \infty$.

Proof. By Delin's Theorem and Proposition 3.3 we have

$$|H_\gamma(z)|e^{-\varphi(z)} \le C|K(z, \bar{\gamma})|e^{-\varphi(z)-\varphi(\gamma)} \le C' e^{-\varepsilon|z-\gamma|},$$

which clearly converges to 0 uniformly on compact sets as $\gamma \to \infty$. Moreover, we have

$$\int_{\mathbb{C}^n} |H_\gamma(z)|^p e^{-p\varphi(z)} d\lambda(z) \lesssim \int_{\mathbb{C}^n} e^{-\varepsilon p|z-\gamma|} d\lambda(z) \le \frac{C}{p^{2n}},$$

and therefore $\{H_\gamma\} \subset \mathscr{F}^p(\varphi)$ is bounded. The modifications for the case $p = \infty$ are straightforward. □

We end with the following convenient proposition.

PROPOSITION 4.3. *Let $p \in [1, \infty]$ and let X be a Banach space. A linear operator $T : \mathscr{F}^p(\varphi) \to X$ is compact if and only if for any sequence $\{f_n\} \subset \mathscr{F}^p(\varphi)$ that is bounded and converges locally uniformly to 0, Tf_n converges to 0 in X.*

Proof. Suppose T is compact and $\{f_n\} \subset \mathscr{F}^p(\varphi)$ is bounded and converges locally uniformly to 0, i.e., $f_n \to 0$ weakly in $\mathscr{F}^p(\varphi)$. If it is not the case that $\|Tf_n\|_X \to 0$, then, by passing to a subsequence if necessary, we may assume there is a $\delta > 0$ such that $\|Tf_n\|_X \geq \delta$. Since T is compact, there is a subsequence $\{f_{n_k}\}$ such that $\|Tf_{n_k} - x\|_X \to 0$ for some $x \in X$. In particular, $Tf_{n_k} \to x$ weakly in X. But for any $\psi \in X^*$, $\psi \circ T \in \mathscr{F}^p(\varphi)^*$ and so

$$\psi(Tf_{n_k}) = (\psi T)(f_{n_k}) \to 0,$$

and thus $Tf_{n_k} \to 0$ weakly. It follows that $x = 0$, which is a contradiction.

Conversely, if T is not compact, there is a sequence $\{g_n\} \subset \mathscr{F}^p(\varphi)$ in the unit ball such that $\{Tg_n\}$ has no convergent subsequence. We claim there exists $g \in \mathscr{F}^p(\varphi)$ and a subsequence g_{n_k} converging weakly to g. Indeed, by Proposition 3.8 (or 3.9 if $p = 1$) $\|\Psi_{g_n}\| \leq \|g_n\| \leq 1$. In other words, $\{\Psi_{g_n}\}$ is a bounded sequence in $\mathscr{F}^q(\varphi)^*$ if $p > 1$ and in $\mathscr{F}^\infty_0(\varphi)^*$ if $p = 1$. By Alaoglu's theorem, there is a subsequence $\{\Psi_{g_{n_k}}\}$ and $\Phi \in \mathscr{F}^q(\varphi)^*$ (resp. $\mathscr{F}^\infty_0(\varphi)^*$) such that $\Psi_{g_{n_k}} \to \Phi$ in the weak-* topology. By Proposition 3.8 or 3.9 there exists $g \in \mathscr{F}^p(\varphi)$ such that $\Phi = \Psi_g$. Then $\Psi_{g_{n_k}}(h) \to \Psi_g(h)$ for all $h \in \mathscr{F}^q(\varphi)$ (resp. $\mathscr{F}^\infty_0(\varphi)$). By Proposition 3.8 (resp. 3.9), $\Phi(g_{n_k}) \to \Phi(g)$ for all $\Phi \in (F^p_\phi)^*$. In other words, $g_{n_k} \to g$ weakly in $\mathscr{F}^p(\varphi)$, as claimed. Therefore the sequence $f_k := g_{n_k} - g \to 0$ weakly. But by construction, Tf_k has no limit in X. This completes the proof. □

5. Geometric Characterization of the Carleson conditions

In this section, let $\varphi \in \mathscr{C}^2(\mathbb{C}^n)$ satisfy $dd^c\varphi \simeq \omega_o$.

5.1. Carleson measures

Recall that, by definition, a positive measure μ is Carleson if the inclusion $\iota_\mu : \mathscr{F}^p(\varphi) \hookrightarrow \mathscr{F}^p(\varphi, \mu)$ is bounded.

THEOREM 5.1 **(Ortega-Cerdà)**. *Let $p \geq 1$ and μ a positive measure in $\mathbb{C}^n$. The following are equivalent.*

(a) *The measure μ is Carleson for $\mathscr{F}^p(\varphi)$.*
(b) *There exists $C > 0$ such that $\mu(B(z,1)) \leq C$ for any $z \in \mathbb{C}^n$.*
(c) *For each $r > 0$ there exists $C_r > 0$ such that $\mu(B(z,r)) \leq C_r$ for any $z \in \mathbb{C}^n$.*

Proof. It is clear that $(b) \iff (c)$. To prove that $(b) \Rightarrow (a)$, cover $\mathbb{C}^n$ by a countable collection of balls of radius 1 such that each point of $\mathbb{C}^n$

is contained in at most N balls, for some fixed number $N \sim 2^n$. On each such ball $B(z,1)$, we have

$$\int_{B(z,1)} |f|^p e^{-p\varphi} d\mu \lesssim \sup_{B(z,1)} |f|^p e^{-p\varphi} \lesssim \int_{B(z,3)} |f|^p e^{-p\varphi} d\lambda,$$

and summing over the countable collection of centers z, we have

$$\begin{aligned} \int_{\mathbb{C}^n} |f|^p e^{-p\varphi} d\mu &\lesssim \sum_j \int_{B(z_j,1)} |f|^p e^{-p\varphi} d\mu \\ &\lesssim \sum_j \int_{B(z_j,3)} |f|^p e^{-p\varphi} d\lambda \\ &\lesssim \int_{\mathbb{C}^n} |f|^p e^{-p\varphi} d\lambda. \end{aligned}$$

Finally we prove $(a) \Rightarrow (b)$. By Proposition 2.5 there is a function $f \in \mathscr{F}^p(\varphi)$ such that $f(z) = e^{\varphi(z)}$ and $||f||_p \leq C$ for some $C > 0$ independent of z. By the estimate (2) of Proposition 2.3 there exists $r > 0$ sufficiently small such that for all $w \in B(z,r)$, $|f(w)|e^{-\varphi(w)} \geq 1/2$. It follows that

$$\begin{aligned} \mu(B(z,r)) &\lesssim 2^p \int_{B(z,r)} |f|^p e^{-p\varphi} d\mu \\ &\leq 2^p \int_{\mathbb{C}^n} |f|^p e^{-p\varphi} d\mu \\ &\lesssim 2^p \int_{\mathbb{C}^n} |f|^p e^{-p\varphi} d\lambda \\ &\lesssim 2^p, \end{aligned}$$

where the Carleson condition is used in the third inequality. The proof is finished. □

5.2. Vanishing Carleson measures

Recall that, by definition, a positive measure μ is vanishing Carleson if the inclusion $\iota_\mu : \mathscr{F}^p(\varphi) \hookrightarrow L^p(e^{-p\varphi} d\mu)$ is compact.

THEOREM 5.2 **(Characterization of Vanishing Carleson measures)**. *Let $p \geq 1$, let $\varphi \in \mathscr{C}^2(\mathbb{C}^n)$ satisfy $dd^c\varphi \simeq \omega_o$, and let μ be a positive measure in $\mathbb{C}^n$. Then the following are equivalent.*

(a) *The measure μ is vanishing Carleson for $\mathscr{F}^p(\varphi)$.*
(b) *For every $\varepsilon > 0$ there exists $R > 0$ such that $\mu(B(z,1)) \leq \varepsilon$ for any $z \in \mathbb{C}^n - B(0,R)$.*

Proof. (b) $\Rightarrow$ (a): First, by (1) of Proposition 2.3 we see that for all $f \in \mathscr{F}^p(\varphi)$

$$|f(z)|^p e^{-p\varphi(z)} \lesssim \int_{\mathbb{C}^n} \mathbf{1}_{B(z,r)} |f(w)|^p e^{-p\varphi(w)} d\lambda(w),$$

and thus

$$\begin{aligned}\int_{\mathbb{C}^n} |f(z)|^p e^{-p\varphi(z)} d\mu(z) &\lesssim \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} \mathbf{1}_{B(z,r)} |f(w)|^p e^{-p\varphi(w)} d\lambda(w) d\mu(z) \\ &= \int_{\mathbb{C}^n} |f(w)|^p e^{-p\varphi(w)} \mu(B(w,r)) d\lambda(w)\end{aligned}$$

Let now $\{f_j\} \subset \mathscr{F}^p(\varphi)$ be a sequence converging weakly to 0. By Proposition 4.1, $||f_j||_{p,\varphi} \leq L$ for some $L > 0$, and $f_j \to 0$ locally uniformly in $\mathbb{C}^n$. By Proposition 4.3 it suffices to show that $f_j \to 0$ in $\mathscr{F}^p(\mu, \varphi)$. To this end, let $\varepsilon > 0$ and choose $R >> 0$ so that $\mu(D(w,1)) < \varepsilon$ for $|w| \geq R$. We therefore have that

$$\begin{aligned}&\int_{\mathbb{C}^n} |f_j(z)|^p e^{-p\varphi(z)} d\mu(z) \\ &\lesssim \int_{B(0,R)} |f_j(w)|^p e^{-p\varphi(w)} \mu(B(w,r)) d\lambda(w) + \varepsilon ||f_j||_{p,\varphi} \\ &\leq 2L^p \varepsilon\end{aligned}$$

provided j is sufficiently large. Thus μ is a vanishing Carleson measure.

(a) $\Rightarrow$ (b): Let $\gamma_n \to \infty$ and set $F_j(z) := K(z, \bar{\gamma}_j) e^{-\varphi(\gamma_j)}$. Then by Delin's Theorem $|F_n(z)| e^{-\varphi(z)} \leq C e^{-\varepsilon|\gamma_n - z|}$ and thus (as we already pointed out in the proof of Proposition 4.2) $||F_j||_{p,\varphi} \leq C$ for some $C > 0$ independent of j. Therefore $F_j \to 0$ locally uniformly. By Proposition 4.3 $||F_j||_{p,\varphi,\mu} \to 0$.

Now, by Proposition 3.3 there exist positive constants C_1, C_2 and ε such that

$$|K(z, \bar{w})| e^{-\varphi(z) - \varphi(w)} \geq C_1 |K(z, \bar{z})| e^{-2\varphi(z)} \geq C_2$$

for all $|z - w| \leq \varepsilon$. We therefore have that

$$\begin{aligned}\int_{\mathbb{C}^n} |F_n(z)|^p e^{-p\varphi(z)} d\mu(z) &= e^{-p\varphi(\gamma_n)} \int_{\mathbb{C}^n} |K(z, \bar{\gamma}_n)|^p e^{-p\varphi(z)} d\mu(z) \\ &\geq \int_{B(\gamma_n, \varepsilon)} |K(z, \bar{\gamma}_n)|^p e^{-p\varphi(\gamma_n)} e^{-p\varphi(z)} d\mu(z) \\ &\geq C\mu(B(\gamma_n, \varepsilon)) |K(\gamma_n, \bar{\gamma}_n)| e^{-2\varphi(\gamma_n)} \\ &\geq C' \mu(B(\gamma_n, \varepsilon)),\end{aligned}$$

and thus $\mu(B(\gamma_n, \varepsilon)) \to 0$ as $\gamma_n \to \infty$. Since there exists a positive integer N such that for any p, $B(p,1)$ is covered by N balls $B(p_1, \varepsilon), ..., B(p_N, \varepsilon)$, whose centers p_i satisfy $|p_i| > |p| - 2$, say, we find that $\mu(B(\gamma_n, 1)) \to 0$ as $\gamma_n \to \infty$. The proof is complete. $\square$

6. Berezin transform

Let μ be a positive measure on $\mathbb{C}^n$. As before, we fix a weight φ such that $dd^c\varphi \simeq \omega_o$. Denote by $K(z, \bar{w})$ the Bergman kernel for $L^2(e^{-2\varphi}d\lambda) \to \mathscr{F}^2(\varphi)$ and let

$$B_z(w) := \frac{K(w, \bar{z})}{\sqrt{K(z, \bar{z})}}.$$

Using the reproducing property and the Hermitian symmetry of the Bergman kernel, we find that

$$\int_{\mathbb{C}^n} |B_z(w)|^2 e^{-2\varphi(w)} d\lambda(w) = 1.$$

DEFINITION 6.1. The function $\tilde{\mu} : \mathbb{C}^n \to [0, +\infty]$ defined by

$$\begin{aligned} \tilde{\mu}(z) &:= \langle T_\mu B_z, B_z \rangle_\varphi \\ &= \int_{\mathbb{C}^n} \left(\int_{\mathbb{C}^n} K(x, \bar{w}) B_z(w) e^{-2\varphi(w)} d\mu(w) \right) \overline{B_z(x)} e^{-2\varphi(x)} d\lambda(x) \end{aligned}$$

is called the Berezin transform of the measure μ. $\diamond$

Again from the reproducing property and the Hermitian symmetry of the Bergman kernel, we see that

$$\tilde{\mu}(z) = \int_{\mathbb{C}^n} |B_z(w)|^2 e^{-2\varphi(w)} d\mu(w). \tag{5}$$

It follows from (5) that for any $\varepsilon > 0$,

$$\int_{B(z,\varepsilon)} |B_z(w)|^2 e^{-2\varphi(w)} d\mu(w) \le \tilde{\mu}(z). \tag{6}$$

6.1. Berezin transforms of Carleson measures

From (5) we see that

$$\tilde{\mu}(z) \le ||\iota_\mu||_{2,\varphi},$$

where $\iota_\mu : \mathscr{F}^2(\varphi) \hookrightarrow \mathscr{F}^2(\mu, \varphi)$ is the inclusion. We therefore conclude that if μ is Carleson for $\mathscr{F}^2(\varphi)$ then $\tilde{\mu}$ is bounded. Of course, by Ortega-Cerdà's Theorem 5.1, we know that μ is Carleson for $\mathscr{F}^2(\varphi)$ if and only if μ is Carleson for $\mathscr{F}^p(\varphi)$. Therefore we have proved that $\tilde{\mu}$ is uniformly bounded when μ is Carleson.

In fact, the converse is also true, namely, if $\tilde{\mu}$ is uniformly bounded then μ is Carleson. Indeed, if $\tilde{\mu}$ is bounded, (6) implies that

$$\int_{B(z,\varepsilon)} |B_z(w)|^2 e^{-2\varphi(w)} d\mu(w) \le C$$

for some positive constant $C > 0$ independent of z. Now, by Proposition 3.3 there exists $c, \varepsilon > 0$ independent of z such that

$$|K(z, \bar{w})| e^{-\varphi(z) - \varphi(w)} \geq c$$

for all $w \in B(z, \varepsilon)$. On the other hand, $K(z, \bar{z}) e^{-2\varphi(z)} \leq c^{-2}$ possibly after shrinking c a little. Thus $|B_z(w)|^2 e^{-2\varphi(w)} \geq c^2$ for all $w \in B(z, \varepsilon)$, and we see that

$$\mu(B(z, \varepsilon)) \leq C_o$$

for some constant C_o independent of z. Therefore μ is Carleson by Ortega-Cerdà's Theorem 5.1. In summary, we have established the following result.

THEOREM 6.2. *Let $\varphi \in \mathscr{C}^2(\mathbb{C}^n)$ be a weight such that $dd^c\varphi \simeq \omega_o$. Then a measure μ is Carleson for $\mathscr{F}^p(\varphi)$ if and only if its Berezin transform $\tilde{\mu}$ is bounded.*

6.2. Berezin transforms of vanishing Carleson measures

Berezin transforms of vanishing Carleson measures are characterized by the following theorem.

THEOREM 6.3. *A measure μ is vanishing Carleson if and only if $\tilde{\mu}$ vanishes at ∞.*

Proof. Suppose μ is vanishing Carleson, meaning that the inclusion map $\iota_\mu : \mathscr{F}^2(\varphi) \to \mathscr{F}^2(\mu, \varphi)$ is compact. It follows that $\iota_\mu(f_n) \to 0$ for any sequence $\{f_n\}$ converging to 0 weakly in $\mathscr{F}^2(\varphi)$. Let $z_n \to \infty$ is $\mathbb{C}^n$ and consider the functions

$$f_n(w) := B_{z_n}(w) = \frac{K(w, \bar{z}_n)}{\sqrt{K(z_n, z_n)}}.$$

We have already seen that $||B_{z_n}||_{2,\varphi} = 1$, and by Delin's Theorem

$$|f_n(w)|^2 = \frac{|K(z_n, \bar{w})|^2 e^{-2\varphi(z_n)}}{K(z_n, z_n) e^{-2\varphi(z_n)}} \leq C e^{2\varphi(w) - \varepsilon|w - z_n|}.$$

Therefore $f_n \to 0$ uniformly on compacts. Thus $\iota_\mu(f_n) \to 0$ in $\mathscr{F}^2(\mu, \varphi)$, i.e.,

$$\tilde{\mu}(z_n) = \int_{\mathbb{C}^n} |f_n(x)|^2 e^{-2\varphi(x)} d\mu(x) \to 0.$$

Conversely, suppose that $\tilde{\mu}$ vanishes at infinity. As in the proof of Theorem 6.2, we have the estimate

$$\mu(B(z, \varepsilon)) \lesssim \tilde{\mu}(z).$$

Therefore by Theorem 5.2 we see that μ is vanishing Carleson. The proof is finished. □

REMARK 6.4. One can define other Berezin-type transforms that achieve the same results. Indeed, the transform

$$B\mu(z) := \int_{\mathbb{C}^n} |K(z,\bar{w})|^2 e^{-2\varphi(z)-2\varphi(w)} d\mu(w)$$

is of course uniformly comparable to $\tilde{\mu}$ by Propositions 3.1 and 3.3. The transform B seems to us to be the simplest choice, but we have gone with the first choice because it was used in past works on the subject. $\diamond$

7. Proofs of the main theorems

7.1. Proof of Theorem 1

Let μ be a Carleson measure. We write

$$k(z,\bar{w}) := |K(z,\bar{w})| e^{-\varphi(z)-\varphi(w)}.$$

As an illustration, we begin with the case $p = 1$. In this case

$$\begin{aligned}
&\int_{\mathbb{C}^n} \left| \int_{\mathbb{C}^n} f(w) K(z,\bar{w}) e^{-2\varphi(w)} d\mu(w) \right| e^{-\varphi(z)} d\lambda(z) \\
&\le \int_{\mathbb{C}^n} \left(\int_{\mathbb{C}^n} |f(w)| e^{-\varphi(w)} k(z,\bar{w}) d\mu(w) \right) d\lambda(z) \\
&= \int_{\mathbb{C}^n} |f(w)| e^{-\varphi(w)} \left(\int_{\mathbb{C}^n} k(z,\bar{w}) d\lambda(z) \right) d\mu(w) \\
&\lesssim \int_{\mathbb{C}^n} |f(w)| e^{-\varphi(w)} d\mu(w) \lesssim \int_{\mathbb{C}^n} |f(w)| e^{-\varphi(w)} d\lambda(w).
\end{aligned}$$

The second inequality follows from Delin's Theorem 3.2, and the last because μ is a Carleson measure.

Turning to $p \in (1, \infty)$, let $f \in \mathscr{F}^p(\varphi)$. Then

$$\begin{aligned}
&\int_{\mathbb{C}^n} \left| \int_{\mathbb{C}^n} f(w) K(z,\bar{w}) e^{-2\varphi(w)} d\mu(w) \right|^p e^{-p\varphi(z)} d\lambda(z) \\
&= \int_{\mathbb{C}^n} \left| \int_{\mathbb{C}^n} f(w) e^{-\varphi(w)} K(z,\bar{w}) e^{-\varphi(z)-\varphi(w)} d\mu(w) \right|^p d\lambda(z) \\
&\le \int_{\mathbb{C}^n} \left(\int_{\mathbb{C}^n} |f(w)| e^{-\varphi(w)} k(z,\bar{w}) d\mu(w) \right)^p d\lambda(z) \\
&= \int_{\mathbb{C}^n} \left(\int_{\mathbb{C}^n} |f(w)| e^{-\varphi(w)} k(z,\bar{w})^{1/p} k(z,\bar{w})^{1/q} d\mu(w) \right)^p d\lambda(z) \\
&\le \int_{\mathbb{C}^n} \left(\int_{\mathbb{C}^n} |f|^p e^{-p\varphi} k(z,\bar{\cdot}) d\mu \right) \left(\int_{\mathbb{C}^n} k(z,\bar{\cdot}) d\mu \right)^{p-1} d\lambda(z).
\end{aligned}$$

Now, since μ is Carleson, Ortega-Cerdà's Theorem 5.1 implies that the inclusion $\iota_\mu : \mathscr{F}^1(\varphi) \to \mathscr{F}^1(\mu, \varphi)$ is bounded. This boundedness may then be applied to $w \mapsto K(w, \bar{z})e^{-\varphi(z)}$ to conclude that

$$\int_{\mathbb{C}^n} k(z, \bar{w}) d\mu(w) \lesssim \int_{\mathbb{C}^n} k(z, \bar{w}) d\lambda(w), \tag{7}$$

and the right side of (7) is finite by Delin's Theorem 3.2. Moreover,

$$\begin{aligned}
&\int_{\mathbb{C}^n} \left(\int_{\mathbb{C}^n} |f(w)|^p e^{-p\varphi(w)} k(z, \bar{w}) d\mu(w) \right) d\lambda(z) \\
&= \int_{\mathbb{C}^n} |f(w)|^p e^{-p\varphi(w)} \left(\int_{\mathbb{C}^n} k(z, \bar{w}) d\lambda(z) \right) d\mu(w) \\
&\lesssim \int_{\mathbb{C}^n} |f|^p e^{-p\varphi} d\mu \\
&\lesssim \int_{\mathbb{C}^n} |f|^p e^{-p\varphi} d\lambda,
\end{aligned}$$

where the second-to-last inequality is again by Delin's Theorem 3.2, and the last inequality is from the definition of Carleson measure.

Finally, we treat the case $p = \infty$. Here we must understand that a measure μ is Carleson by definition if $\mu(B(z, 1)) \leq C$ for some z-independent constant C. Assuming μ is Carleson in this sense, we see that if $f \in \mathscr{F}^\infty(\varphi)$ then

$$\begin{aligned}
&\sup_{z \in \mathbb{C}^n} \left| \int_{\mathbb{C}^n} f(w) K(z, \bar{w}) e^{-2\varphi(w)} d\mu(w) \right| e^{-\varphi(z)} \\
&\leq ||f||_\infty \sup_{z \in \mathbb{C}^n} \int_{\mathbb{C}^n} k(z, \bar{w}) d\mu(w) \\
&\lesssim ||f||_\infty \sup_{z \in \mathbb{C}^n} \int_{\mathbb{C}^n} k(z, \bar{w}) d\lambda(w) \lesssim ||f||_\infty .
\end{aligned}$$

The second inequality is of course (7). Thus $T_\mu : \mathscr{F}^p(\varphi) \to \mathscr{F}^p(\varphi)$ is well-defined and bounded if μ is Carleson.

Conversely, suppose $T_\mu : \mathscr{F}^p(\varphi) \to \mathscr{F}^p(\varphi)$ is bounded. By the reproducing property of the Bergman kernel we have the equality

$$K(z, \bar{w}) = \int_{\mathbb{C}^n} K(x, \bar{w}) K(z, \bar{x}) e^{-2\varphi(x)} d\lambda(x).$$

Using this, we have from Proposition 3.3 that

$$
\begin{aligned}
&\mu(B(z,\varepsilon))\\
&\lesssim \int_{B(z,\varepsilon)} |K(z,\bar{w})|^2 e^{-2(\varphi(z)+\varphi(w))} d\mu(w)\\
&\leq \int_{\mathbb{C}^n} K(w,\bar{z})e^{-(\varphi(z)+\varphi(w))} K(z,\bar{w})e^{-(\varphi(z)+\varphi(w))} d\mu(w)\\
&= \int_{\mathbb{C}^n} \Bigg[K(w,\bar{z})e^{-(\varphi(z)+\varphi(w))}\\
&\qquad \left(\int_{\mathbb{C}^n} K(x,\bar{w})K(z,\bar{x})e^{-2\varphi(x)} d\lambda(x)\right) e^{-(\varphi(z)+\varphi(w))}\Bigg] d\mu(w)\\
&= \int_{\mathbb{C}^n} \Bigg[\left(\int_{\mathbb{C}^n} K(w,\bar{z})e^{-\varphi(z)} K(x,\bar{w})e^{-2\varphi(w)} d\mu(w)\right)\\
&\qquad \overline{K(x,\bar{z})e^{-\varphi(z)}} e^{-2\varphi(x)}\Bigg] d\lambda(x)\\
&= \left(T_\mu(K(\cdot,\bar{z})e^{-\varphi(z)}), K(\cdot,\bar{z})e^{-\varphi(z)}\right)\\
&\leq ||T_\mu K(\cdot,\bar{z})e^{-\varphi(z)}||_p ||K(\cdot,\bar{z})e^{-\varphi(z)}||_q,
\end{aligned}
$$

and the rightmost term is bounded independent of z by Delin's Theorem 3.2 and the boundedness of T_μ. The proof of Theorem 1 is finished. □

7.2. Proof of Theorem 2

Suppose first that T_μ is compact. Let $F_z(w) := K(w,\bar{z})e^{-\varphi(z)}$ be as in Proposition 4.2. Then $F_z \to 0$ weakly in $\mathscr{F}^p(\varphi)$ as $z \to \infty$, and thus by Hölder's Inequality we have

$$
|\langle T_\mu F_z, F_z\rangle| \leq ||T_\mu F_z||_{\varphi,p} ||F_z||_{\varphi,q} \to 0
$$

as $z \to \infty$. On the other hand,

$$
\begin{aligned}
&|\langle T_\mu F_z, F_z\rangle|\\
&= \int_{\mathbb{C}^n}\int_{\mathbb{C}^n} K(x,\bar{w})F_z(w)e^{-2\varphi(w)}\overline{F_z(x)}e^{-2\varphi(x)} d\mu(w)d\lambda(x)\\
&= \int_{\mathbb{C}^n} |F_z(w)|^2 e^{-2\varphi(w)} d\mu(w)\\
&\geq \int_{B(z,1)} |F_z(w)|^2 e^{-2\varphi(w)} d\mu(w)\\
&\geq C\mu(B(z,1)),
\end{aligned}
$$

and therefore by Theorem 5.2 μ is a vanishing Carleson measure.

Conversely, suppose μ is vanishing Carleson. As in the Proof of Theorem 1, if $f \in \mathscr{F}^p(\varphi)$ then

$$\begin{aligned}
&\int_{\mathbb{C}^n} |T_\mu f|^p e^{-p\varphi} d\lambda \\
&= \int_{\mathbb{C}^n} \left| \int_{\mathbb{C}^n} f(w) e^{-\varphi(w)} k(z,w) d\mu(w) \right|^p d\lambda(z) \\
&\le \int_{\mathbb{C}^n} \left(\int_{\mathbb{C}^n} |f|^p e^{-p\varphi} k(z,\bar{\cdot}) d\mu \right) \left(\int_{\mathbb{C}^n} k(z,\bar{\cdot}) d\mu \right)^{p-1} d\lambda(z) \\
&\lesssim \int_{\mathbb{C}^n} \left(\int_{\mathbb{C}^n} |f(w)|^p e^{-p\varphi(w)} k(z,\bar{w}) d\mu(w) \right) d\lambda(z) \\
&\le C \int_{\mathbb{C}^n} |f(w)|^p e^{-p\varphi(w)} d\mu(w).
\end{aligned}$$

(As before, we use the Carleson condition applied to the holomorphic function $w \mapsto K(w,\bar{z}) e^{-\varphi(z)}$ as well as Delin's Theorem.) In other words, $||T_\mu|| \le C||\iota_\mu||$, where $\iota_\mu : \mathscr{F}^p(\varphi) \to \mathscr{F}^p(\mu,\varphi)$ is the inclusion map. Since ι_μ is compact by hypothesis, T_μ is also compact. The proof of Theorem 2 is finished. □

REMARK 7.1. Another proof of the compactness of T_μ when μ is vanishing Carleson proceeds as follows. First, for $f \in \mathscr{F}^p(\varphi)$ and $g \in \mathscr{F}^q(\varphi) \cap \mathscr{F}^2(\varphi)$ we have

$$\begin{aligned}
\langle T_\mu f, g \rangle &= \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} f(w) K(z,\bar{w}) e^{-2\varphi(w)} d\mu(w) \overline{g(z)} e^{-2\varphi(z)} d\lambda(z) \\
&= \int_{\mathbb{C}^n} f(w) \overline{g(w)} d\mu(w) = \langle f, T_\mu g \rangle
\end{aligned}$$

This means, with the density of $\mathscr{F}^p(\varphi) \cap \mathscr{F}^2(\varphi)$ taken into account, that T_μ is compact if and only if it is weak*-compact. But then if $f_n \to 0$ weakly, then the identity $\langle T_\mu f_n, g \rangle = \langle f_n, T_\mu g \rangle$ shows that $T_\mu f_n \to 0$ weakly, and therefore by Proposition 4.3 T_μ is weak*-compact. □

Appendix A. Berndtsson's twisted $\bar{\partial}$ estimates and Delin's Theorem

A.1. Review of the $\bar{\partial}$-Neumann problem

Let Ω be a pseudoconvex domain with smooth boundary in $\mathbb{C}^n$, and let $\varphi \in \mathscr{C}^2$ be a smooth weight. Assume θ is a $(0,1)$-form in $L^2(\lambda,\varphi)$ such that $\bar{\partial}\theta = 0$. If there is a solution u of the equation $\bar{\partial} u = \theta$ such that

$$\int_\Omega |u|^2 e^{-\varphi} d\lambda < +\infty,$$

then there is a solution whose L^2-norm is minimal. Our goal will be to estimate this minimal solution.

Since any two solutions u_1 and u_2 must differ by a holomorphic function, the solution u_o of minimal norm must be orthogonal to the holomorphic functions on Ω. It follows, if we ignore the delicate issue of whether $\bar{\partial}^*$ has closed range (which is the case for the situation we will consider below), that any solution u of minimal norm is of the form

$$u = \bar{\partial}^* \beta$$

for some $(0,1)$-form β, where $\bar{\partial}^*$ denotes the formal adjoint of $\bar{\partial}$. Note that since u has minimal norm, it is unique. Indeed, if u' is another function satisfying $\bar{\partial} u' = \theta$ then, since $u - u'$ is holomorphic and thus orthogonal to u,

$$||u'||^2 = ||(u' - u) + u||^2 = ||u - u'||^2 + ||u||^2.$$

Thus $||u|| = ||u'||$ implies that $u = u'$.

Knowing that a solution is minimal does not say anything directly about the size of the solution. The key to getting explicit estimates for the minimal solution u lies in obtaining estimates for the form β. Indeed,

$$||u||^2 = (\bar{\partial}\bar{\partial}^* \beta, \beta) \le ||\theta|| \cdot ||\beta||,$$

so an estimate on β would clearly give an estimate on u. Of course, while the solution u of minimal norm is unique, the form β such that $u = \bar{\partial}^* \beta$ is clearly not unique. To choose the β of minimal norm, we should therefore choose β such that satisfies

$$\bar{\partial}\bar{\partial}^* \beta = \theta \quad \text{and} \quad \beta \perp \text{Kernel}(\bar{\partial}^*).$$

Since $\text{Kernel}(\bar{\partial}^*)_{p,q} \supset \text{Image}(\bar{\partial}^*_{p,q+1})$, the second condition implies that $\bar{\partial}\beta = 0$, and therefore β satisfies the equation

$$\Box \beta = \theta,$$

where $\Box = \bar{\partial}\bar{\partial}^* + \bar{\partial}^* \bar{\partial}$ is the so-called *Kohn Laplacian*, *i.e.*, the Laplace-Beltrami operator associated to $\bar{\partial}$. Conversely, if β satisfies $\Box \beta = \theta$ then

$$||\bar{\partial}^* \bar{\partial} \beta||^2 = (\bar{\partial}(\bar{\partial}^* \bar{\partial}\beta), \bar{\partial}\beta) = (\bar{\partial}\Box\beta, \bar{\partial}\beta) = (\bar{\partial}\theta, \bar{\partial}\beta) = 0,$$

and thus

$$||\bar{\partial}\beta||^2 = (\bar{\partial}^* \bar{\partial}\beta, \beta) = 0.$$

One solves the equation $\Box\beta = \theta$ by way of the so-called *Basic Identity*: if $\rho : \mathbb{C}^n \to \mathbb{R}$ is a smooth function such that $\Omega = \rho^{-1}(-\infty, 0)$, $\partial\Omega = \rho^{-1}(0)$ and $||d\rho|| \equiv 1$ on $\partial\Omega$, then

$$||\bar{\partial}^* \alpha||^2 + ||\bar{\partial}\alpha||^2 = \int_\Omega \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j} \alpha_{\bar{i}} \overline{\alpha_{\bar{j}}} e^{-\varphi} d\lambda + ||\overline{\nabla}\alpha||^2 + \int_{\partial\Omega} \frac{\partial^2 \rho}{\partial z^i \partial \bar{z}^j} \alpha_{\bar{i}} \overline{\alpha_{\bar{j}}} e^{-\varphi} d\sigma \tag{8}$$

for all smooth $(0,1)$-forms $\alpha = \sum_j \alpha_{\bar{j}} d\bar{z}^j$ satisfying the so-called $\bar{\partial}$-*Neumann boundary condition*

$$\sum \frac{\partial \rho}{\partial z^i} \alpha_{\bar{i}} \bigg|_{\partial\Omega} \equiv 0.$$

Here σ is the surface area measure on $\partial\Omega$.

REMARK A.1. For completeness of exposition, we note that

$$\nabla \left(\sum_j \alpha_{\bar{j}} d\bar{z}^j \right) = \sum_{j,k} \frac{\partial \alpha_{\bar{j}}}{\partial \bar{z}^k} d\bar{z}^k \otimes d\bar{z}^j.$$

We will not, however, use this formula. Only the fact that $||\overline{\nabla}\alpha|| \geq 0$ will matter to us. $\diamond$

The Basic Identity implies that if Ω is pseudoconvex (so that the last integral on the right side of (8) is non-negative) and $dd^c\varphi \geq c\omega_o$, then for all smooth α in the domain of $\bar{\partial}^*$,

$$(\Box\alpha, \alpha) \geq c||\alpha||^2. \tag{9}$$

The inequality (9) says that $\Box$ has an inverse. Making this last statement precise relies on the delicate point of showing that the smooth forms satisfying the $\bar{\partial}$-Neumann boundary condition are dense in the domain of the densely defined Hilbert space adjoint of $\bar{\partial}$. If we accept this density, a proof of which the reader can find in Hörmander's paper [H-1965], the invertibility is easily proved as follows. First, we define the subspace H of $L^2(e^{-\varphi}d\lambda)$ as the Hilbert space closure of all the smooth forms α such that $||\bar{\partial}^*\alpha||^2 + ||\bar{\partial}\alpha||^2 < +\infty$. The basic estimate (9) says that these forms are in $L^2(e^{-\varphi}d\lambda)$ and that the inclusion $H \hookrightarrow L^2(e^{-\varphi}d\lambda)$ is bounded. Now let $\alpha \in L^2(e^{-\varphi}d\lambda)$, and define the anti-linear functional

$$\lambda(\alpha) := (\theta, \alpha).$$

Then

$$|\lambda(\alpha)|^2 \leq ||\alpha||^2||\theta||^2 \leq \frac{||\theta||^2}{c}(||\bar{\partial}^*\alpha||^2 + ||\bar{\partial}\alpha||^2).$$

It follows that $\lambda \in H^*$, and thus by the Riesz Representation Theorem there exists $\beta \in H$ such that

$$||\beta||_H^2 = ||\lambda||_{H^*}^2 \leq \frac{||\theta||^2}{c} \quad \text{and} \quad (\theta, \alpha) = (\bar{\partial}^*\beta, \bar{\partial}^*\alpha) + (\bar{\partial}\beta, \bar{\partial}\alpha).$$

The latter equation says that $\Box\beta = \theta$. Moreover

$$||u||^2 = ||\bar{\partial}^*\beta||^2 = ||\beta||_H^2 \leq \frac{||\theta||^2}{c}.$$

REMARK A.2. A more difficult argument can be used to show that the solution β of the equation $\Box\beta = \theta$ is smooth up to the boundary when this is the case for θ. This smoothness is certainly the most important and difficult part of the solution of the $\bar\partial$-Neumann problem by Kohn. $\diamond$

REMARK A.3. Finally, note that since $\beta \in H$, β lies in the domain of $\bar\partial^*$. In particular, since β is smooth, it satisfies the $\bar\partial$-Neumann boundary condition. $\diamond$

A.2. The twisted basic identity

Our next goal will be to estimate the minimal solution u in ways other than with respect to the L^2-norm. The key tool is the twisted basic identity, which we now describe.

Consider a new weight $e^{-\psi}$ defined as follows: let τ be a positive smooth function and set

$$e^{-\varphi} = \tau e^{-\psi}.$$

Note that since $\bar\partial^*_\psi \alpha = -\sum_i e^{\psi}\frac{\partial(e^{-\psi}\alpha_{\bar i})}{\partial z^i}$, we have the formula

$$\bar\partial^*_\varphi \alpha = \bar\partial^*_\psi \alpha - \frac{1}{\tau}\sum_i \frac{\partial \tau}{\partial z^i}\alpha_{\bar i}.$$

Moreover

$$\partial\bar\partial\varphi = \partial\bar\partial\psi - \partial\bar\partial\log\tau = \partial\bar\partial\psi - \frac{\partial\bar\partial\tau}{\tau} + \frac{\partial\tau\wedge\bar\partial\tau}{\tau^2}.$$

Substituting these two formulas into the Basic Identity (8) gives, after some straightforward manipulation and replacing ψ with φ, the so-called *Twisted Basic Identity*:

$$\begin{aligned} ||\sqrt{\tau}\bar\partial^*_\varphi\alpha||^2 + ||\sqrt{\tau}\bar\partial\alpha||^2 \ = \ & \int_\Omega \left(\tau\frac{\partial^2\varphi}{\partial z^i\partial\bar z^j} - \frac{\partial^2\tau}{\partial z^i\partial\bar z^j}\right)\alpha_{\bar i}\overline{\alpha_{\bar j}}e^{-\varphi}d\lambda \\ & +2\mathrm{Re}\int_\Omega \bar\partial^*_\varphi\alpha\overline{\frac{\partial\tau}{\partial z^i}\alpha_{\bar i}}e^{-\varphi}d\lambda \qquad (10) \\ & +||\sqrt{\tau}\overline{\nabla}\alpha||^2 + \int_{\partial\Omega}\tau\frac{\partial^2\rho}{\partial z^i\partial\bar z^j}\alpha_{\bar i}\overline{\alpha_{\bar j}}e^{-\varphi}d\sigma. \end{aligned}$$

Here again, this identity holds for all smooth forms in the domain of $\bar\partial^*_\varphi$, as the reader can verify.

A.3. Berndtsson's Estimates for the minimal solution

We are now ready to prove the following theorem [B-1997].

THEOREM A.4 **(Berndtsson)**. *Let $\tau : \Omega \to (0,\infty)$ be a $\mathscr{C}^2$-function and let A be a symmetric matrix whose entries are functions in Ω such that at* *each point $z \in \Omega$, $A(z)$ is positive definite. Assume furthermore that the matrix*

$$\left(\tau\frac{\partial^2\varphi}{\partial z^i\partial\bar{z}^j} - \frac{\partial^2\tau}{\partial z^i\partial\bar{z}^j} - \tau A_{i\bar{j}}\right)$$

is positive-semi-definite at each point of Ω. Then the solution u of $\bar{\partial}u = \theta$ having minimal norm in $L^2(e^{-\varphi}d\lambda)$ satisfies the estimate

$$\int_\Omega \tau|u|^2 e^{-\varphi}d\lambda \leq \int_\Omega \tau|\theta|_A^2 e^{-\varphi}d\lambda.$$

where

$$|\theta|_A^2 = \sum_{i\bar{j}}(A^{-1})_{i\bar{j}}\theta_{\bar{i}}\overline{\theta_{\bar{j}}}.$$

Proof. Integration-by-parts shows that

$$\int_\Omega \bar{\partial}_\varphi^*\alpha\overline{\sum_i \frac{\partial\tau}{\partial z^i}\alpha_{\bar{i}}}e^{-\varphi}d\lambda = -\int_\Omega \tau\sum_i(\bar{\partial}\bar{\partial}_\varphi^*\alpha)_{\bar{i}}\overline{\alpha_{\bar{i}}}e^{-\varphi} + ||\sqrt{\tau}\bar{\partial}_\varphi^*\alpha||^2,$$

(there is no boundary term because α satisfies the $\bar{\partial}$-Neumann boundary conditions) and thus (10) becomes

$$\begin{aligned} 2\mathrm{Re}\int_\Omega \tau\left\langle\bar{\partial}\bar{\partial}_\varphi^*\alpha,\alpha\right\rangle e^{-\varphi}d\lambda + ||\sqrt{\tau}\bar{\partial}\alpha||^2 &= ||\sqrt{\tau}\bar{\partial}_\varphi^*\alpha||^2 \\ &+ \int_\Omega\left(\tau\frac{\partial^2\varphi}{\partial z^i\partial\bar{z}^j} - \frac{\partial^2\tau}{\partial z^i\partial\bar{z}^j}\right)\alpha_{\bar{i}}\overline{\alpha_{\bar{j}}}e^{-\varphi}d\lambda + ||\sqrt{\tau}\bar{\nabla}\alpha||^2 \\ &+ \int_{\partial\Omega}\tau\frac{\partial^2\rho}{\partial z^i\partial\bar{z}^j}\alpha_{\bar{i}}\overline{\alpha_{\bar{j}}}e^{-\varphi}d\sigma. \end{aligned} \tag{11}$$

(The identity (11) is a twisted version of what has been called the $\partial\bar{\partial}$-*Bochner-Kodaira Identity* by Siu in [S-1982].) We now apply the identity (11) to the form β such that $\Box\beta = \theta$. Recalling that $\bar{\partial}\beta = 0$ and that $u = \bar{\partial}^*\beta$ solves $\bar{\partial}u = \theta$, we have

$$\begin{aligned} &2\mathrm{Re}\int_\Omega \tau\langle\theta,\beta\rangle e^{-\varphi}d\lambda \\ &= ||\sqrt{\tau}u||^2 + \int_\Omega\left(\tau\frac{\partial^2\varphi}{\partial z^i\partial\bar{z}^j} - \frac{\partial^2\tau}{\partial z^i\partial\bar{z}^j}\right)\beta_{\bar{i}}\overline{\beta_{\bar{j}}}e^{-\varphi}d\lambda \\ &+ ||\sqrt{\tau}\bar{\nabla}\beta||^2 + \int_{\partial\Omega}\tau\frac{\partial^2\rho}{\partial z^i\partial\bar{z}^j}\beta_{\bar{i}}\overline{\beta_{\bar{j}}}e^{-\varphi}d\sigma \end{aligned} \tag{12}$$

From the identity (12), the inequality

$$2\mathrm{Re}\ \langle\theta,\beta\rangle \leq \sum_{i,j}A_{i\bar{j}}\beta_i\overline{\beta_j} + |\theta|_A^2$$

and the pseudoconvexity of Ω imply the estimate

$$\int_\Omega \tau|\theta|_A^2 e^{-\varphi}d\lambda \geq \int_\Omega \tau|u|^2e^{-\varphi}d\lambda + \int_\Omega \sum_{i\bar{j}} \left(\tau\frac{\partial^2\varphi}{\partial z^i\partial\bar{z}^j} - \frac{\partial^2\tau}{\partial z^i\partial\bar{z}^j} - \tau A_{i\bar{j}}\right)\beta_i\overline{\beta_{\bar{j}}}e^{-\varphi}d\lambda.$$

The hypothesis implies that the right-most integral is non-negative, and thus the proof is complete. □

A.4. Proof of Theorem 3.2

Fix two points z and w in $\mathbb{C}^n$. In view of Proposition 3.1 we may assume that $|z-w| \geq 4$. We fix once and for all a smooth function $\chi \in \mathscr{C}_0^\infty(B(w,2))$ such that $0 \leq \chi \leq 1$, $\chi|_{B(w,1)} \equiv 1$ and $|\bar\partial\chi| \leq 2$ pointwise.

Using (1) of Proposition 2.3, we have

$$\begin{aligned}
&|K(z,\bar{w})|e^{-\varphi(w)-\varphi(z)} \\
&\quad\lesssim \int_{B(w,1)} |K(z,\bar\zeta)|e^{-\varphi(\zeta)-\varphi(z)}d\lambda(\zeta) \\
&\quad\lesssim \left(\int_{B(w,1)} |K(z,\bar\zeta)|^2e^{-2\varphi(\zeta)}d\lambda(\zeta)\right)^{1/2} e^{-\varphi(z)} \\
&\quad\leq \left(\int_{\mathbb{C}^n} |K(z,\bar\zeta)|^2\chi(\zeta)e^{-2\varphi(\zeta)}d\lambda(\zeta)\right)^{1/2} e^{-\varphi(z)} \\
&\quad= \left(\int_{\mathbb{C}^n} (\chi(\zeta)K(\zeta,\bar{z}))\,K(z,\bar\zeta)e^{-2\varphi(\zeta)}d\lambda(\zeta)\right)^{1/2} e^{-\varphi(z)} \\
&\quad= |P(\chi K(\cdot,\bar{z})e^{-\varphi(z)})(z)|^{1/2}e^{-\frac{1}{2}\varphi(z)},
\end{aligned}$$

where P is the Bergman projection. Now, the function $\zeta \mapsto \chi(\zeta)K(\zeta,\bar{z})e^{-\varphi(z)}$ is smooth compactly supported, and therefore it is in the domain of $\bar\partial$. It follows that, since $\chi(z) = 0$,

$$P(\chi K(\cdot,\bar{z})e^{-\varphi(z)})(z) = \chi(z)K(z,\bar{z})e^{-\varphi(z)} - u(z) = -u(z),$$

where u is the solution of the equation $\bar\partial u = \bar\partial(\chi K(\cdot,\bar{z})e^{-\varphi(z)})$ having minimal L^2-norm. Moreover, since $\chi \equiv 0$ on $B(z,1)$, $u \in \mathcal{O}(\overline{B(z,1)})$ and therefore again by (1) of Proposition 2.3,

$$|u(z)|^2e^{-2\varphi(z)} \lesssim \int_{B(z,1)} |u|^2e^{-2\varphi}d\lambda.$$

Let $\tau(\zeta) := e^{-\varepsilon|\zeta-z|}$. Then for $\varepsilon > 0$ sufficiently small, τ satisfies the hypotheses of Berndtsson's Theorem A.4 with $A_{i\bar{j}} = c\delta_{i\bar{j}}$ for some small constant c. Indeed,

$$\sqrt{-1}\partial\bar\partial\tau = \varepsilon^2\tau\sqrt{-1}(\partial|\zeta - z|)\wedge(\bar\partial|\zeta - z|) - \varepsilon\tau\sqrt{-1}\partial\bar\partial|\zeta - z| \le \varepsilon^2\tau\omega_o,$$

and since $dd^c\varphi \simeq \omega_o$,

$$\tau dd^c\varphi - dd^c\tau - c\tau\omega_o \ge 0.$$

(This computation was done in [B-1997].) It follows that

$$\begin{aligned} |u(z)|^2 e^{-2\varphi(z)} &\lesssim \int_{B(z,1)} |u|^2 e^{-2\varphi} d\lambda \lesssim \int_{B(z,1)} |u|^2 \tau e^{-2\varphi} d\lambda \\ &\lesssim \int_{\mathbb{C}^n} \tau |\bar\partial\chi|^2 |K(\cdot,\bar z)|^2 e^{-2\varphi(z)-2\varphi(\zeta)} d\lambda(\zeta). \end{aligned}$$

Observe finally that $\bar\partial\chi$ is supported on $B(w,2)$, and for $\zeta \in B(w,2)$, $|\zeta - z| \ge |w - z| - 2$. Therefore on $B(w,2)$,

$$\tau \le e^{2\varepsilon - \varepsilon|z-w|}.$$

Since $|K(\zeta,\bar z)|e^{-\varphi(\zeta)-\varphi(z)} \lesssim 1$ by Proposition 3.1, we obtain the estimate

$$|u(z)|^2 e^{-2\varphi(z)} \lesssim e^{-\varepsilon|w-z|},$$

and therefore

$$|K(z,\bar w)|e^{-\varphi(z)-\varphi(w)} \le Ce^{-\varepsilon_o|z-w|}$$

for some constants C and ε_o that do not depend on z and w. This is precisely what we wanted to prove. □

A.5. Berndtsson's Uniform $\bar\partial$ Theorem

Finally, we use the following theorem of Berndtsson.

THEOREM A.5. *Suppose $\varphi \in \mathscr{C}^2(\mathbb{C}^n)$ satisfies $C^{-1}\omega_o \le dd^c\varphi \le C\omega_o$. Then there is a constant A, depending only on C and not on φ, such that the for every $\bar\partial$-closed $(0,1)$-form $\theta = \sum_{i,\bar j=1}^n \theta_{i\bar j} dz^i \wedge d\bar z^j$ satisfying*

$$||\theta||_{\infty,\varphi} := \sup_{z\in\mathbb{C}^n} \sum_{i,\bar j=1}^n |\theta_{i\bar j}(z)|e^{-\varphi(z)} < +\infty$$

and

$$||\theta||^2_{2,\varphi} := \int_{\mathbb{C}^n} \sum_{i,\bar j=1}^n |\theta_{i\bar j}|^2 e^{-2\varphi} d\lambda < +\infty$$

the minimal solution $u \in L^2(e^{-2\varphi}d\lambda)$ of the equation $\bar\partial u = \theta$ satisfies the estimate

$$\sup_{\mathbb{C}^n} |u|e^{-\varphi} \le A||\theta||_{\infty,\varphi}.$$

The result in [B-1997] is more general, considering φ for which only the lower bound $dd^c\varphi \geq C\omega_o$ holds. However, for the more general result a certain plurisubharmonic capacity associated to φ appears in the estimate, and this capacity is uniformly bounded when the curvature of the weight is also bounded above by a multiple of the Euclidean metric.

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