Minischool on $L^2$ Extension
Lecture 3: Sharp Constants

Dror Varolin
Stony Brook University

Workshop on $L^2$ Extension
University of Tokyo
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1. Berndtsson’s Direct Image Theorem

2. Suita Conjecture

3. Proof of Ohsawa-Takegoshi by Berndtsson-Lempert
Berndtsson’s Direct Image Theorem

**X complex manifold**

$L \rightarrow X \times \mathbb{D}$, $e^{-\varphi}$ be a holomorphic line bundle with s.h.m.

Assume: $\forall \tau \in \mathbb{D}$, $e^{-\varphi \tau} := e^{-\varphi}|_{X \tau}$ is s.h.m. on the fiber $X \tau := X \times \{\tau\}$.

Fix a volume form $dV$ on $X$.

On each fiber $X \tau$ one has an $L^2$ structure for sections of $L \tau := L|_{X \tau}$:

$$||f||^2_\tau := \int_{X \tau} |f|^2 e^{-\varphi \tau} dV.$$ 

The Hilbert space

$$\mathcal{H}^2_\tau := \{f \in H^0(X \tau, L \tau) ; ||f||_\tau < +\infty\}$$

is then a fiber of a Hilbert fiber bundle $\mathcal{H}^2 \rightarrow \mathbb{D}$.

If we further constrain $X$ and $e^{-\varphi}$ (for example, if $X$ is a bounded domain in some larger manifold $Y$, and if $e^{-\varphi}$ is the restriction of some smooth metric on $Y \times \mathbb{C}$, say) then this bundle is a locally trivial vector bundle over $\mathbb{D}$ with a non-trivial metric, which we assume from here on.

Berndtsson established a version of the following result for domains in $\mathbb{C}^n$. His proof carries over to the case of Stein manifolds.
Let $X$ be a complex manifold and $L 	o X \times \mathbb{D}$ be a holomorphic line bundle with s.h.m. Assume: $\forall \tau \in \mathbb{D}$, $e^{-\varphi_{\tau}} := e^{-\varphi}|_{X_{\tau}}$ is s.h.m. on the fiber $X_{\tau} := X \times \{\tau\}$. Fix a volume form $dV$ on $X$. On each fiber $X_{\tau}$ one has an $L^2$ structure for sections of $L_{\tau} := L|_{X_{\tau}}$:

$$||f||^2_{\tau} := \int_{X_{\tau}} |f|^2 e^{-\varphi_{\tau}} dV.$$ 

The Hilbert space $\mathcal{H}_{\tau}^2 := \{f \in H^0(X_{\tau}, L_{\tau}) ; ||f||_{\tau} < +\infty\}$ is then a fiber of a Hilbert fiber bundle $\mathcal{H}^2 \to \mathbb{D}$. If we further constrain $X$ and $e^{-\varphi}$ (for example, if $X$ is a bounded domain in some larger manifold $Y$, and if $e^{-\varphi}$ is the restriction of some smooth metric on $Y \times \mathbb{C}$, say) then this bundle is a locally trivial vector bundle over $\mathbb{D}$ with a non-trivial metric, which we assume from here on. Berndtsson established a version of the following result for domains in $\mathbb{C}^n$. His proof carries over to the case of Stein manifolds.
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Theorem (Direct Image Theorem: Open Case)

Let $X$ be a bounded pseudoconvex domain in a Stein manifold $Y$ with a volume form $dV$ and $L \to Y \times \mathbb{D}$ a holomorphic line bundle with smooth Hermitian metric $e^{-\varphi}$. Assume that

$$\sqrt{-1} (\partial \bar{\partial} \varphi - \varphi^* \partial \bar{\partial} \log dV)$$

is a positive (resp. non-negative) $(1, 1)$-form on $X \times \mathbb{D}$, where $\varphi : Y \times \mathbb{D} \to Y$ is the projection to the first factor. Then the Hilbert bundle $\mathcal{H}^2 \to \mathbb{D}$, with its $L^2$ metric $\| \cdot \|_\tau$, $\tau \in \mathbb{D}$, has positive (resp. non-negative) curvature.

If $\mathbb{D}$ replaced with $\mathbb{B}$, Berndtsson proved $\mathcal{H}^2 \to \mathbb{B}$ Nakano-positive. Nakano positivity is the strongest notion of positivity. Implies the weaker but somewhat more natural Griffiths positivity:
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Implies the weaker but somewhat more natural Griffiths positivity:
A Hilbert bundle \( \mathcal{H} \to B \) over a complex base manifold \( B \) is said to be Griffiths positive (resp. non-negative) if for any holomorphic section \( \xi \) of the dual Hilbert bundle \( \mathcal{H}^* \to B \) the function

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b \mapsto ||\xi(b)||_b
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is strictly plurisubharmonic (resp. plurisubharmonic).

When the base manifold is 1-dimensional, Nakano and Griffiths positivity agree.
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**Theorem (Berndtsson)**

*Y Stein manifold, $L \to Y \times \mathbb{D}$, $e^{-\varphi}$ Hermitian holo line bundle.*

Write $e^{-\varphi_\tau} := e^{-\varphi}|_{Y \times \{\tau\}}$.

Let $X \subset Y$ be a pseudoconvex domain.

Consider

$$\mathcal{H}_\tau^2 := \left\{ \alpha \in H^0(Y, K_X \otimes L_{X \times \{\tau\}}) ; \int_X |\alpha|^2 e^{-\varphi_\tau} < +\infty \right\}.$$

Let $B_\tau$ be the Bergman kernel of $\mathcal{H}_\tau^2$, i.e.,

$$B_\tau(z, \bar{z}) = \sum_j (-1)^{n^2/2} \alpha_j^{(\tau)}(z) \wedge \overline{\alpha_j^{(\tau)}(z)},$$

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THEOREM (BERNDTSSON)

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Theorem (Berndtsson (ctd))

Then for each \( z_0 \in X \), any nowhere-zero holomorphic section \( \xi \) of \( L \to U \times \mathbb{D} \), and any nowhere-zero holomorphic top-form \( \beta \), in a sufficiently small neighborhood \( U \) of \( z_0 \) in \( X \), the function

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(z, \tau) \mapsto \log \frac{B_\tau(z, \bar{z}) \otimes \xi(z, \tau) \otimes \overline{\xi(z, \tau)}}{(-1)^{n^2/2} \beta(z) \wedge \overline{\beta(z)}}
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is plurisubharmonic on \( U \times \mathbb{D} \).

Proof.

Since this function (1) is obviously plurisubharmonic in \( z \) for any fixed \( \tau \), it suffices to prove that it is subharmonic in \( \tau \) for any fixed \( z \). Let \( z \in U \). Consider the linear functional \( \delta_z : \mathcal{H}_t^{2} \to \mathbb{C} \) defined by

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\langle \delta_z(\tau), \alpha \rangle = \frac{\alpha(z) \otimes \xi(z, \tau)}{\beta(z)}, \quad \alpha \in \mathcal{H}_T^{2}.
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Proof (ctd).

$\delta_z(\tau): \mathcal{H}_\tau^{\mathbb{C}} \to \mathbb{C}$ is point evaluation, hence bounded.

The section $\delta_z$ is holomorphic (in $\tau$) because $\xi(z, \cdot)$ is holomorphic.

Finally, the norm of $\delta_z(\tau)$ is

$$||\delta_z(\tau)||_{\mathcal{H}_\tau^{\mathbb{C}}}^2 = \sup_{\alpha \in \mathcal{H}_\tau^{\mathbb{C}}, ||\alpha||=1} \frac{\alpha(z) \wedge \overline{\alpha(z)} \otimes \xi(z, \tau) \otimes \overline{\xi(z, \tau)}}{\beta(z) \wedge \overline{\beta(z)}}$$

$$= \frac{B_\tau(z, \bar{z}) \otimes \xi(z, \tau) \otimes \overline{\xi(z, \tau)}}{(-1)^n/2 \beta(z) \wedge \overline{\beta(z)}}$$

Thus the result follows from Berndtsson’s Direct Image Theorem. \qed
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\|\delta_z(\tau)\|_{*\tau}^2 = \sup_{\alpha \in \mathcal{H}_\tau^2, \|\alpha\|_{\tau} = 1} \frac{\alpha(z) \wedge \overline{\alpha(z)} \otimes \xi(z, \tau) \otimes \overline{\xi(z, \tau)}}{\beta(z) \wedge \overline{\beta(z)}}
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\[
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Thus the result follows from Berndtsson’s Direct Image Theorem. \( \square \)
Proof (ctd).

\( \delta_z(\tau) : \mathcal{H}_\tau^2 \to \mathbb{C} \) is point evaluation, hence bounded. The section \( \delta_z \) is holomorphic (in \( \tau \)) because \( \xi(z, \cdot) \) is holomorphic. Finally, the norm of \( \delta_z(\tau) \) is

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A version of Berndtsson’s Direct Image Theorem also holds when the total space is a family that is not a product.

In this more general setting, there are complications arising from the fact that the associated Hilbert bundle is not locally trivial.

A $\bar{\partial}$ operator may nevertheless be defined on sections that are holomorphic in the fiber. This $\bar{\partial}$ operator paves the way for defining a Chern connection as the unique connection (in this setting) that is compatible with the Hermitian fiber metrics and has $\bar{\partial}$ for its $(0, 1)$-part.

However, the $\bar{\partial}$ operator is not well-defined on sections that are not holomorphic along the fiber, so the Griffiths curvature formula approach of Berndtsson cannot be extended to the family setting.

Fortunately, the curvature of this not-necessarily locally trivial Hilbert bundle, defined as in the case of a vector bundle, can be calculated directly. This was done recently by Xu Wang.
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**Remark**

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In particular, the theorem paves the way for degeneration arguments in complex analytic geometry.

Berndtsson has applied such a degeneration technique to prove the openness conjecture.

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Berndtsson’s Direct Image Theorem

Suita Conjecture

Proof of Ohsawa-Takegoshi by Berndtsson-Lempert
To demonstrate the degeneration technique, we give another proof of the Suita conjecture.

Let $X$ be a Riemann surface admitting a Green’s function $G$. Fix $x \in X$ and choose $f \in \mathcal{O}(X)$ such that $\text{Ord}(f) = \{x\}$. We have our fundamental metric

$$
\omega_F(x) = e^{2h_X(x)} \frac{\sqrt{-1}}{2} df(x) \wedge d\bar{f}(x),
$$

where

$$
h_X(x) := \lim_{y \to x} G(x, y) - \log |f(y)|.
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As we explained in Lecture 2, the goal is to estimate

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One can define the Riemann surfaces

\[ X_t := \{ y \in X ; \ G(x, y) < t \}, \quad t < 0. \]

The function \( G(x, y) - \log|f(y)| \) is harmonic (so smooth) on \( X \), and limits to \( h_X(x) \) as \( y \to x \). Therefore for \( t \ll 0 \),

\[ X_t \approx \left\{ y \in X ; \ |f(y)| < e^t e^{-h_X(x)} \right\} = f(D_r(0)) \subset \mathbb{C}, \quad r = e^{t-h_X(x)}. \]

For \( D_r(0) \), Bergman kernel (with our normalization) is

\[ B_{D_r(0)}(z, \bar{z}) = \frac{2}{\pi r^2} \frac{\sqrt{-1}dz \wedge d\bar{z}}{\left(1 - \frac{|z|^2}{r^2}\right)^2}. \]

Thus for \( t \) sufficiently large and negative,

\[ B_{X_t}(x) \approx \frac{4e^{2h_X(x)-2t}}{\pi} \frac{\sqrt{-1}}{2} df(x) \wedge d\bar{f}(x) = \frac{4e^{-2t}}{\pi} \omega_F. \]

In fact:

\[ \lim_{t \to -\infty} \frac{e^{2t} B_{X_t}(x, \bar{x})}{\omega_F(x)} = \frac{4}{\pi}. \]
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The most important step in the proof is to prove the following theorem.

**Theorem**

The function

\[ t \mapsto \log \frac{B_{X_t}(x, \bar{x})}{\sqrt{-1} df(x) \wedge d\bar{f}(x)} \]

is convex.

- This result is quite close to the theorem of Berndtsson on the plurisubharmonicity of the Bergman kernel in the product case.
- Nevertheless, we cannot use that theorem in our proof, because the underlying spaces were all the same, and only the weights varied.
- The Direct Image Theorem of Xu Wang, which deals with the case of a non-trivial family, can be used in place of Berndtsson’s.
- However, we will use Berndtsson’s Direct Image Theorem and certain weights that grow rapidly on \( X \setminus X_t \), and obtain the desired result as a “limiting case”.

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**THEOREM**

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- However, we will use Berndtsson’s Direct Image Theorem and certain weights that grow rapidly on $X - X_t$, and obtain the desired result as a “limiting case”.
We define the auxiliary plurisubharmonic weight function 
\( \psi : X \times \mathbb{H} \to [-\infty, 0] \) by

\[
\psi(z, \tau) := \max(2G_x(z) - \text{Re} \, \tau, 0),
\]

where \( \mathbb{H} := \{ \tau \in \mathbb{C} ; \text{Re} \, \tau < 0 \} \) denotes the left half plane.

For \( p > 0 \), we define

\[
\mathcal{H}^2_{\tau, (p)} := \left\{ \alpha \in H^0(X, K_X) ; \| \alpha \|^2_{\tau, p} := \int_X |\alpha|^2 e^{-p\psi(\tau, \cdot)} < +\infty \right\}.
\]

Note that, as \( p \to \infty \), the square-norm \( \| \alpha \|^2_{\tau, p} \) converges to

\[
\| \alpha \|^2_{\tau} := \int_{X \text{Re} \, \tau} |\alpha|^2.
\]

Our convexity theorem will be proved if

\[
\lim_{p \to \infty} \frac{B_{t, (p)}(x, \bar{x})}{\sqrt{-1} df(x) \wedge d\bar{f}(x))} = \frac{B_{X_t}(x, \bar{x})}{\sqrt{-1} df(x) \wedge d\bar{f}(x))}.
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\[ \lim_{p \to \infty} \frac{B_{t, (p)}(x, \bar{x})}{\sqrt{-1} df(x) \wedge d\bar{f}(x))} = \frac{B_{X,t}(x, \bar{x})}{\sqrt{-1} df(x) \wedge d\bar{f}(x)} \]

Dror Varolin (Stony Brook)
We define the auxiliary plurisubharmonic weight function
\( \psi : X \times \mathbb{H} \to [-\infty, 0] \) by
\[
\psi(z, \tau) := \max(2G_X(z) - \Re \tau, 0),
\]
where \( \mathbb{H} := \{ \tau \in \mathbb{C} ; \Re \tau < 0 \} \) denotes the left half plane.
For \( p > 0 \), we define
\[
\mathcal{H}^2_{\tau, (p)} := \left\{ \alpha \in H^0(X, K_X) ; \| \alpha \|_{\tau, p}^2 := \int_X |\alpha|^2 e^{-p\psi(\tau, \cdot)} < +\infty \right\}.
\]
Note that, as \( p \to \infty \), the square-norm \( \| \alpha \|_{\tau, p}^2 \) converges to
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Our convexity theorem will be proved if

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From the extremal characterization of the Bergman kernel one sees that
\[
\frac{B_{t,(p)}(x, \bar{x})}{\sqrt{-1}df(x) \wedge d\bar{f}(x))} \leq \frac{B_{X_t}(x, \bar{x})}{\sqrt{-1}df(x) \wedge d\bar{f}(x))}.
\]

On the other hand, let \(\alpha_t \in H^0(X_t, K_{X_t})\) be the unique holomorphic 1-form such that
\[
\int_{X_t} |\alpha_t|^2 = 1 \quad \text{and} \quad B_{X_t}(x, \bar{x}) = \sqrt{-1}\alpha_t(x) \wedge \overline{\alpha_t(x)}.
\]

If \(p >> 0\), then by a somewhat technical, but believable, cutoff-correct technique, we can find \(\tilde{\alpha}_t \in \mathcal{H}_{t,(p)}^2\) with
\[
\tilde{\alpha}_t(x) = \alpha_t(x) \quad \text{and} \quad ||\tilde{\alpha}_t||^2_{t,p} \leq 1 + \varepsilon.
\]

For arbitrarily small \(\varepsilon > 0\). Therefore
\[
\frac{B_{t,(p)}(x, \bar{x})}{\sqrt{-1}df(x) \wedge d\bar{f}(x))} \geq \frac{1}{1 + \varepsilon} \frac{|\alpha_t(x)|^2}{|df(x)|^2} = \frac{B_{X_t}(x, \bar{x})}{(1 + \varepsilon)\sqrt{-1}df(x) \wedge d\bar{f}(x))}.
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Thus we have proved the convexity theorem.
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\]
Thus we have proved the convexity theorem.
End of the second proof of the Suita conjecture.

By the convexity theorem for Bergman kernels, the function

\[ k : t \mapsto \frac{e^{2t} B_{X_t}(x, \bar{x})}{\omega_F(x)} \]

is convex. Since it is also bounded below on the negative real axis, it must be increasing. In particular, we find that

\[ \frac{B_X(x, \bar{x})}{\omega_F(x)} = k(0) \geq \lim_{t \to -\infty} \frac{e^{2t} B_{X_t}(x, \bar{x})}{\omega_F(x)} = \frac{4}{\pi}. \]
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1. Berndtsson’s Direct Image Theorem

2. Suita Conjecture

3. Proof of Ohsawa-Takegoshi by Berndtsson-Lempert
In order to focus on the main idea of the use of degeneration techniques in $L^2$ extension, let us prove the earliest version of the extension theorem, namely the theorem on $L^2$ extension from a hyperplane in a bounded pseudoconvex domain.

Let $(\zeta, z) \in \mathbb{C} \times \mathbb{C}^{n-1}$ be coordinates in $\mathbb{C}^n$.

Fix a pseudoconvex $X \subset \subset \mathbb{C}^n = \mathbb{C} \times \mathbb{C}^{n-1}$, and let $Z = \{\zeta = 0\} \cap X$.

Let us define the Hilbert spaces

$$\mathcal{H}_\varphi^2 := \left\{ F \in \mathcal{O}(X) : \int_X |F|^2 e^{-\varphi} dV < +\infty \right\}$$

and

$$\mathcal{S}_\varphi^2 := \left\{ f \in \mathcal{O}(Z) : \int_Z |f|^2 e^{-\varphi} dA < +\infty \right\}.$$

To prove the Extension Theorem, we need to construct an extension operator $\mathcal{S}_\varphi^2 \to \mathcal{H}_\varphi^2$ with a universal bound on its norm. Therefore, we might as well assume we are working with the operator of minimal extensions.
In order to focus on the main idea of the use of degeneration techniques in $L^2$ extension, let us prove the earliest version of the extension theorem, namely the theorem on $L^2$ extension from a hyperplane in a bounded pseudoconvex domain.

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Fix a pseudoconvex $X \subset \mathbb{C}^n = \mathbb{C} \times \mathbb{C}^{n-1}$, and let $Z = \{\zeta = 0\} \cap X$.

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$$H^2_\varphi := \left\{ F \in \mathcal{O}(X) ; \int_X |F|^2 e^{-\varphi} dV < +\infty \right\}$$

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To prove the Extension Theorem, we need to construct an extension operator $\mathcal{H}_\varphi^2 \to H_\varphi^2$ with a universal bound on its norm. Therefore, we might as well assume we are working with the operator of minimal extensions.
In order to focus on the main idea of the use of degeneration techniques in \( L^2 \) extension, let us prove the earliest version of the extension theorem, namely the theorem on \( L^2 \) extension from a hyperplane in a bounded pseudoconvex domain. Let \((\zeta, z) \in \mathbb{C} \times \mathbb{C}^{n-1}\) be coordinates in \(\mathbb{C}^n\). Fix a pseudoconvex \( X \subset \subset \mathbb{C}^n = \mathbb{C} \times \mathbb{C}^{n-1} \), and let \( Z = \{ \zeta = 0 \} \cap X \). Let us define the Hilbert spaces

\[
H^2_\varphi := \left\{ F \in \mathcal{O}(X) ; \int_X |F|^2 e^{-\varphi} dV < +\infty \right\}
\]

and

\[
\tilde{H}^2_\varphi := \left\{ f \in \mathcal{O}(Z) ; \int_Z |f|^2 e^{-\varphi} dA < +\infty \right\}.
\]

To prove the Extension Theorem, we need to construct an extension operator \( \tilde{H}^2_\varphi \to H^2_\varphi \) with a universal bound on its norm. Therefore, we might as well assume we are working with the operator of minimal extensions.
In order to focus on the main idea of the use of degeneration techniques in $L^2$ extension, let us prove the earliest version of the extension theorem, namely the theorem on $L^2$ extension from a hyperplane in a bounded pseudoconvex domain. Let $(\zeta, z) \in \mathbb{C} \times \mathbb{C}^{n-1}$ be coordinates in $\mathbb{C}^n$. Fix a pseudoconvex $X \subset \subset \mathbb{C}^n = \mathbb{C} \times \mathbb{C}^{n-1}$, and let $Z = \{\zeta = 0\} \cap X$. Let us define the Hilbert spaces

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If at the end of the day we obtain a universal bound, then the usual limiting arguments of complex analysis allow us to make certain reductions: We may assume that

- $X$ is (smoothly) bounded pseudoconvex domain,
- $\varphi$ is smooth on a neighborhood of $\overline{X}$ in $\mathbb{C}^n$, and
- $Z \cap \partial X$.

Also enough to extend the elements of the dense subspace

$$\mathcal{S} := \mathcal{H}_\varphi^2(Z) \cap \mathcal{O}(\overline{X}).$$

We denote by $\mathcal{I}_Z$ the ideal sheaf of germs of holomorphic functions on $X$ that vanish on $Z$.

Then $\mathcal{O}_Z \cong \mathcal{O}_X/\mathcal{I}_Z$, so by Steinness, there is an isomorphism

$$H^0(Z, \mathcal{O}_Z) \cong H^0(X, \mathcal{O}_X)/H^0(X, \mathcal{I}_Z).$$

Via this isomorphism and the Hilbert space structure for $\mathcal{H}_\varphi^2$, we obtain a second norm on $\mathcal{H}_\varphi^2$:

$$\|f\|_o^2 := \min \left\{ \int_X |F|^2 e^{-\varphi} dV ; \ F \in \mathcal{H}_\varphi^2 \text{ and } F|_Z = f \right\}.$$
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Via this isomorphism and the Hilbert space structure for $\mathcal{H}^2$, we obtain a second norm on $\mathcal{H}^2$:

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Via this isomorphism and the Hilbert space structure for $\mathcal{H}_\varphi^2$, we obtain a second norm on $\mathcal{H}_\varphi^2$:

$$\|f\|_{\mathcal{O}}^2 := \min \left\{ \int_X |F|^2 e^{-\varphi} dV ; F \in \mathcal{H}_\varphi^2 \text{ and } F|_Z = f \right\}.$$
If finite, $||f||_o$ is the norm of the unique minimal extension of $f$ to $X$, namely, the extension that is orthogonal to

$$\mathcal{F}^2_Z(X) := H^0(X, I_Z) \cap H^2_\varphi(X).$$

In fact, this norm is finite on the aforementioned dense subspace $\mathcal{G}$. We want to find the best constant $C_o$ such that

$$||f||_o^2 \leq C_o \int_X |f|^2 e^{-\varphi} dA, \quad f \in \mathcal{G}.$$ 

If we denote by $F_o$ the unique element of $H^2_\varphi \ominus \mathcal{F}^2_Z(X)$ extending $f$,

$$||f||_o^2 = \sup \left\{ ||\xi(F_o)||/||\xi||_* ; \xi \in H^2_\varphi, \xi|_{\mathcal{F}^2_Z(X)} \equiv 0. \right\}.$$ 

The presence of $||\xi||_*$ is the link to Berndtsson’s Theorem.
If finite, $\|f\|_o$ is the norm of the unique minimal extension of $f$ to $X$, namely, the extension that is orthogonal to

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In fact, this norm is finite on the aforementioned dense subspace $\mathcal{S}$. We want to find the best constant $C_o$ such that

$$\|f\|^2_o \leq C_o \int_Z |f|^2 e^{-\varphi} dA, \quad f \in \mathcal{S}.$$

If we denote by $F_o$ the unique element of $\mathcal{H}_\varphi^2 \oplus \mathcal{J}_Z^2(X)$ extending $f$,

$$\|f\|^2_o = \sup \left\{ |\xi(F_o)|/\|\xi\|_* ; \xi \in \mathcal{H}_\varphi^{2*}, \xi|_{\mathcal{J}_Z^2(X)} \equiv 0 \right\}.$$

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In fact, this norm is finite on the aforementioned dense subspace $\mathcal{G}$. We want to find the best constant $C_o$ such that

$$\|f\|_o^2 \leq C_o \int_Z |f|^2 e^{-\varphi} dA, \quad f \in \mathcal{G}.$$ 

If we denote by $F_o$ the unique element of $\mathcal{H}_\varphi^2 \ominus \mathcal{J}_Z^2(X)$ extending $f$,

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\]

If we denote by \( F_o \) the unique element of \( \mathcal{H}_\phi^2 \ominus \mathcal{J}^2_Z(X) \) extending \( f \),

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The approach is analogous to the idea just used to prove Suita:

- If the domain $X$ is an almost infinitesimal neighborhood of $Z$, then any extension has finite norm.
- We degenerate $X$ to $Z$ through a family $X_t$, $t \leq 0$, where $X_0 = X$ and $X_t \to Z$ as $t \to -\infty$. Denoting the best constant by $C_o(t)$, we want to show that $C_o(t)$ is decreasing. As we will see, the crucial ingredient to prove this decreasing property is the Direct Image Theorem.

As in the case of the proof of the Suita Conjecture, we cannot degenerate the actual domains, because Berndtsson’s Theorem, which treats only trivial families, will no longer apply. (However, one might use Wang.)

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Let $T(\zeta, z) := \frac{\zeta}{\text{diam}(X)}$, and define the non-negative function

$$\phi_t := \max(0, \log |T|^2 - t), \quad t < 0.$$ 

Now define the Hilbert spaces

$$\mathcal{H}^2_{p,t} := \left\{ F \in \mathcal{O}(X) ; \int_X |F|^2 e^{-p\phi_t - \psi} dV < +\infty \right\}$$

and

$$\mathcal{H}^2 := \left\{ f \in \mathcal{O}(Z) ; \int_Z |f|^2 e^{-\psi} dA < +\infty \right\}.$$ 

Observe that

$$\lim_{p \to \infty} \int_X |F|^2 e^{-p\phi_t - \psi} dV = \int_{X_t} |F|^2 e^{-\psi} dV,$$

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Now fix the function $f \in \mathcal{O}(\overline{Z})$ to be extended. Let $F_t \in H_t^2$ denote the extension of minimal norm, i.e., the extension of $f$ that is orthogonal to $\mathcal{J}_t^2$. The norm of $F_t$ is given by the dual formula

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\|f\|_o = \|F_t\|_t = \sup \left\{ \frac{|\langle \xi, F_t \rangle_t|}{\|\xi\|_{H_t^{2*}}} ; \xi \in H_t^{2*} \text{ and } \xi|_{\mathcal{J}_t^2} \equiv 0 \right\}.
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To compute this norm, it suffices to take the supremum over a dense subspace of the space of the annihilator of $\mathcal{J}_t^2$ in $H_t^{2*}$. 
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The set of functionals
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\left\{ \xi_g : F \mapsto \int_Z F \overline{g} e^{-\psi} dA \ ; \ g \in \mathcal{C}_0^\infty(Z) \right\}
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is such a dense subspace, since
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\xi_g(F) = 0 \text{ for all } g \in \mathcal{C}_0^\infty(Z) \implies F|_Z \equiv 0.
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where \( \tilde{g} \) is the Bergman projection, of \( g \) onto \( \mathcal{H}^2 \). Thus
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\|f\|_0^2 = \|F_t\|_t^2 \leq \left( \sup_{g \in \mathcal{C}_0^\infty(Z)} \|\xi_g\|_{t^*}^{-2} \int_Z |\tilde{g}|^2 e^{-\psi} dA \right) \int_Z |f|^2 e^{-\psi} dA.
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**Proposition**

The function

\[ t \mapsto \log \| \xi_g \|^2_{t^*} \]

is convex.

*Proof.*

Berndtsson’s Direct Image Theorem.

**Proposition**

One has the bound

\[ \| \xi_g \|^2_{t^*} e^t = O(1), \quad t \sim -\infty. \]

Consequently,

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Proof.
The convexity of $k_g$ follows from previous proposition.
A bounded convex function on $(-\infty, 0]$ must be increasing, so the consequence follows from the bound.
To establish the bound, observe that for $t << 0$, the submean-value property and the smoothness of $\psi$ imply that for any $h \in H^2_t$,

$$\int_{Z \cap \text{supp}(g)} |h|^2 e^{-\psi} dA \leq \tilde{C} e^{-t} \int_{X_t} |h|^2 e^{-\psi} dA \leq \tilde{C} e^{-t} ||h||^2_t.$$ 

Thus

$$||\xi_g||^2_{t^*} = \sup_{||h||_t=1} \int_Z h\bar{g} e^{-\psi} dA \leq \tilde{C} e^{-t} \int_Z |g|^2 e^{-\psi} dA =: Ce^{-t},$$

as claimed.
Proof.

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To establish the bound, observe that for $t << 0$, the submean-value property and the smoothness of $\psi$ imply that for any $h \in \mathcal{H}_t^2$,

$$\int_{Z \cap \text{supp}(g)} |h|^2 e^{-\psi} dA \leq \tilde{C} e^{-t} \int_{\chi_t} |h|^2 e^{-\psi} dA \leq \tilde{C} e^{-t} ||h||_t^2.$$

Thus

$$||\xi g||_{t^*}^2 = \sup_{||h||_t=1} \int_Z h \bar{g} e^{-\psi} dA \leq \tilde{C} e^{-t} \int_Z |g|^2 e^{-\psi} dA =: C e^{-t},$$

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Thus

$$\|\xi_g\|_{t*}^2 = \sup_{\|h\|_t=1} \int_Z h\bar{g} e^{-\psi} dA \leq \tilde{C} e^{-t} \int_Z |g|^2 e^{-\psi} dA =: C e^{-t},$$

as claimed.
Proof.

The convexity of $k_g$ follows from previous proposition. A bounded convex function on $(-\infty, 0]$ must be increasing, so the consequence follows from the bound.

To establish the bound, observe that for $t \ll 0$, the submean-value property and the smoothness of $\psi$ imply that for any $h \in \mathcal{H}_t^2$,

$$
\int_{Z \cap \text{supp}(g)} |h|^2 e^{-\psi} \, dA \leq \tilde{C} e^{-t} \int_{X_t} |h|^2 e^{-\psi} \, dA \leq \tilde{C} e^{-t} \|h\|^2_t.
$$

Thus

$$
\|\xi_g\|^2_{t*} = \sup_{\|h\|_t = 1} \int_Z h \bar{g} e^{-\psi} \, dA \leq \tilde{C} e^{-t} \int_Z |g|^2 e^{-\psi} \, dA =: C e^{-t},
$$

as claimed.
**Proposition**

\[
\lim_{t \to -\infty} \|\xi g\|_{t^*}^2 e^t \geq \frac{1}{\pi (\text{diam}(X))^2} \int_Z |\tilde{g}|^2 e^{-\psi} dA.
\]

**Proof:**

First, after shrinking the initial $X$ again, one can assume that there is an $\mathcal{H}^2$-approximation of $\tilde{g}$ by the restriction to $Z$ of a holomorphic function $\hat{g} \in \mathcal{O}(\overline{X})$. Then for any $\varepsilon > 0$, if we choose a sufficiently good approximation $\hat{g}$, we have

\[
\|\xi g\|_{t^*} \geq \left| \int_Z \hat{g} \overline{\tilde{g}} e^{-\psi} dA \right| / \|\hat{g}\|_t \geq (1 - \varepsilon) \|\tilde{g}\|^2 / \|\hat{g}\|_t.
\]

Now,

\[
\|\hat{g}\|^2_t = \int_{X_t} |\hat{g}|^2 e^{-\psi} dA + \int_{\log |T|^2 > t} |\hat{g}|^2 e^{-\psi - p(t - \log |T|^2)} dA.
\]
**Proposition**

\[
\lim_{t \to -\infty} \left\| \xi g \right\|_{t^*}^2 e^t \geq \frac{1}{\pi (\text{diam}(X))^2} \int_Z |\tilde{g}|^2 e^{-\psi} dA.
\]

**Proof:**
First, after shrinking the initial \( X \) again, one can assume that there is an \( \mathcal{H}^2 \)-approximation of \( \tilde{g} \) by the restriction to \( Z \) of a holomorphic function \( \hat{g} \in \mathcal{O}(\overline{X}) \). Then for any \( \varepsilon > 0 \), if we choose a sufficiently good approximation \( \hat{g} \), we have

\[
\left\| \xi g \right\|_{t^*} \geq \left| \int_Z \hat{g} \tilde{g} e^{-\psi} dA \right| / \left\| \hat{g} \right\|_t \geq (1 - \varepsilon) \left\| \tilde{g} \right\|^2 / \left\| \hat{g} \right\|_t.
\]

Now,

\[
\left\| \hat{g} \right\|^2_t = \int_{X_t} |\hat{g}|^2 e^{-\psi} dA + \int_{\log |T|^2 > t} |\hat{g}|^2 e^{-\psi + p(t - \log |T|^2)} dA.
\]
**Proposition**

\[
\lim_{t \to -\infty} \|\xi g\|_{t_*}^2 e^t \geq \frac{1}{\pi (\text{diam}(X))^2} \int_Z |\tilde{g}|^2 e^{-\psi} dA.
\]

**Proof:**
First, after shrinking the initial \(X\) again, one can assume that there is an \(\mathcal{S}^2\)-approximation of \(\tilde{g}\) by the restriction to \(Z\) of a holomorphic function \(\hat{g} \in \mathcal{O}(X)\). Then for any \(\varepsilon > 0\), if we choose a sufficiently good approximation \(\hat{g}\), we have

\[
\|\xi g\|_{t_*} \geq \left| \int_Z \hat{g} \tilde{g} e^{-\psi} dA \right| / \|\hat{g}\|_t \geq (1 - \varepsilon) \|\tilde{g}\|^2 / \|\hat{g}\|_t.
\]

Now,

\[
\|\hat{g}\|_t^2 = \int_{X_t} |\hat{g}|^2 e^{-\psi} dA + \int_{\log |T|^2 > t} |\hat{g}|^2 e^{-\psi + p(t - \log |T|^2)} dA.
\]
Then for $t << 0$,

$$\int_{X_t} |\hat{g}|^2 e^{-\psi} dA \leq (1 + \varepsilon)\pi (\text{diam}(X))^2 e^t \int_Z |\hat{g}|^2 e^{-\psi} dA$$

$$\leq (1 + \varepsilon)^2 \pi (\text{diam}(X))^2 e^t ||\tilde{g}||^2;$$

On the other hand, since $\hat{g}$ and $\psi$ both extend to a neighborhood of $\overline{X}$, we have

$$\int_{\log |T|^2 > t} |\hat{g}|^2 e^{-\psi + p(t - \log |T|^2)} dA \leq ||\hat{g}e^{-\psi/2}||^2_{\infty} \int_{\log |T|^2 > t} e^{p(t - \log |T|^2)} dA.$$

By Fubini we have

$$\int_{\log |T|^2 > t} |\hat{g}|^2 e^{-\psi + p(t - \log |T|^2)} dA \leq C_0 \int_t^0 e^{p(t-s)} V'(s) ds,$$

where $V(t)$ denotes the volume of $X_t$, which is continuous, piecewise smooth, increasing, and smooth outside some compact subset of $(-\infty, 0]$. Moreover,

$$\lim_{t \to -\infty} e^{-t} V(t) = A_0 \text{ exists, and } V(t) \leq C e^t. \quad (2)$$
Then for $t << 0$,

$$
\int_{X_t} |\hat{g}|^2 e^{-\psi} dA \leq (1 + \varepsilon)\pi (\text{diam}(X))^2 e^t \int_Z |\hat{g}|^2 e^{-\psi} dA
$$

$$
\leq (1 + \varepsilon)^2 \pi (\text{diam}(X))^2 e^t \|\tilde{g}\|^2;
$$

On the other hand, since $\hat{g}$ and $\psi$ both extend to a neighborhood of $X$, we have

$$
\int_{\log |T|^2 > t} |\hat{g}|^2 e^{-\psi + \rho(t - \log |T|^2)} dA \leq \|\hat{g} e^{-\psi/2}\|^2_{\infty} \int_{\log |T|^2 > t} e^{\rho(t - \log |T|^2)} dA.
$$

By Fubini we have

$$
\int_{\log |T|^2 > t} |\hat{g}|^2 e^{-\psi + \rho(t - \log |T|^2)} dA \leq C_0 \int_0^t e^{\rho(t-s)} V'(s) ds,
$$

where $V(t)$ denotes the volume of $X_t$, which is continuous, piecewise smooth, increasing, and smooth outside some compact subset of $(-\infty, 0]$. Moreover,

$$
\lim_{t \to -\infty} e^{-t} V(t) = A_0 \text{ exists, and } V(t) \leq C e^t. \quad (2)
$$
Then for $t \ll 0$, 
\[
\int_{X_t} |\hat{g}|^2 e^{-\psi} \, dA \leq (1 + \varepsilon)\pi \text{diam}(X)^2 e^t \int_Z |\hat{g}|^2 e^{-\psi} \, dA \\
\leq (1 + \varepsilon)^2 \pi \text{diam}(X)^2 e^t \|\tilde{g}\|^2;
\]
On the other hand, since $\hat{g}$ and $\psi$ both extend to a neighborhood of $X$, we have
\[
\int_{\log |T|^2 > t} |\hat{g}|^2 e^{-\psi + p(t - \log |T|^2)} \, dA \leq \|\hat{g}e^{-\psi/2}\|_{\infty}^2 \int_{\log |T|^2 > t} e^{p(t - \log |T|^2)} \, dA.
\]
By Fubini we have
\[
\int_{\log |T|^2 > t} |\hat{g}|^2 e^{-\psi + p(t - \log |T|^2)} \, dA \leq C_0 \int_t^0 e^{p(t - s)} V'(s) \, ds,
\]
where $V(t)$ denotes the volume of $X_t$, which is continuous, piecewise smooth, increasing, and smooth outside some compact subset of $(-\infty, 0]$. Moreover,
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\]
Then for $t << 0$,

$$ \int_{\mathcal{X}_t} |\hat{g}|^2 e^{-\psi} dA \leq (1 + \varepsilon)\pi (\text{diam}(X))^2 e^t \int_Z |\hat{g}|^2 e^{-\psi} dA$$

$$\leq (1 + \varepsilon)^2 \pi (\text{diam}(X))^2 e^t ||\tilde{g}||^2;$$

On the other hand, since $\hat{g}$ and $\psi$ both extend to a neighborhood of $\overline{X}$, we have

$$\int_{\log |T|^2 > t} |\hat{g}|^2 e^{-\psi + p(t-\log |T|^2)} dA \leq ||\hat{g}e^{-\psi/2}||_\infty^2 \int_{\log |T|^2 > t} e^{p(t-\log |T|^2)} dA.$$  

By Fubini we have

$$\int_{\log |T|^2 > t} |\hat{g}|^2 e^{-\psi + p(t-\log |T|^2)} dA \leq C_0 \int_0^t e^{p(t-s)} V'(s) ds,$$

where $V(t)$ denotes the volume of $X_t$, which is continuous, piecewise smooth, increasing, and smooth outside some compact subset of $(-\infty, 0]$. Moreover,

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Then for $t << 0$,

$$\int_{X_t} |\hat{g}|^2 e^{-\psi} \, dA \leq (1 + \varepsilon) \pi (\text{diam}(X))^2 e^t \int_Z |\hat{g}|^2 e^{-\psi} \, dA$$

$$\leq (1 + \varepsilon)^2 \pi (\text{diam}(X))^2 e^t \|\tilde{g}\|^2;$$

On the other hand, since $\hat{g}$ and $\psi$ both extend to a neighborhood of $\overline{X}$, we have

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By Fubini we have

$$\int_{\log |T|^2 > t} |\hat{g}|^2 e^{-\psi + p(t - \log |T|^2)} \, dA \leq C_0 \int_t^0 e^{p(t-s)} V'(s) \, ds,$$

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\leq (1 + \varepsilon)^2 \pi (\text{diam}(X))^2 e^t ||\tilde{g}||^2;
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\[
\lim_{t \to -\infty} e^{-t} V(t) = A_0 \text{ exists, and } V(t) \leq C e^t. \quad (2)
\]
We claim that the integral
\[ f_p(t) := e^{-t} \int_t^0 e^{p(t-s)} V'(s) \, ds \]
can be made as small as we like, provided we take \(|t|\) and \(p\) sufficiently large.

To see this claim, observe that
\[
\begin{align*}
f_p(t) &= e^{(p-1)t} \int_T^0 e^{-ps} V'(s) \, ds + e^{-t} \int_T^T e^{p(t-s)} V'(s) \, ds \\
&\leq e^{(p-1)t-pT} (V(T) - V(0)) + e^{-t} \int_T^T e^{p(t-s)} V'(s) \, ds.
\end{align*}
\]

The first term is made as small as we like by taking \(t \ll T\).
We claim that the integral
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\]
\[
 \leq e^{(p-1)t-pT} (V(T) - V(0)) + e^{-t} \int_t^T e^{p(t-s)} V'(s) \, ds.
\]
The first term is made as small as we like by taking $t << T$. 
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& \leq e^{(p-1)t-pT} (V(T) - V(0)) + e^{-t} \int_t^T e^{p(t-s)} V'(s) ds.
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Dror Varolin  (Stony Brook)
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&\leq e^{(p-1)t-pT} (V(T) - V(0)) + e^{-t} \int_{t}^{T} e^{p(t-s)} V'(s) ds.
\end{align*}
\]

The first term is made as small as we like by taking \(t << T\).
\[ e^{-t} \int_t^T e^{p(t-s)} V'(s) ds \]

\[ \overset{(IBP)}{=} V(0)e^{(p-1)t} - e^{-t} V(t) + pe^{pt} \int_t^T e^{-ps} V(s) ds \]

\[ = V(0)e^{(p-1)t} - e^{-t} V(t) + pe^{(p-1)t} \int_t^T e^{-(p-1)s} e^{-s} V(s) ds \]

\[ \leq V(0)e^{(p-1)t} + e^{-t} V(t) \left( -1 + pe^{(p-1)t} \int_t^T e^{-(p-1)s} ds \right) \]

\[ + \delta pe^{(p-1)t} \int_t^T e^{-(p-1)s} dt \]

\[ = V(0)e^{(p-1)t} + e^{-t} V(t) \left( -1 + \frac{p}{p-1} (1 - e^{(p-1)(t-T)}) \right) \]

\[ + \delta \frac{p}{p-1} (1 - e^{(p-1)(t-T)}). \]
\[ e^{-t} \int_t^T e^{p(t-s)} V'(s) ds \]

\[(IBP) \quad V(0) e^{(p-1)t} - e^{-t} V(t) + pe^{pt} \int_t^T e^{-ps} V(s) ds \]

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\[+ \delta pe^{(p-1)t} \int_t^T e^{-(p-1)s} dt \]

\[= V(0) e^{(p-1)t} + e^{-t} V(t) \left( -1 + \frac{p}{p-1} \left( 1 - e^{(p-1)(t-T)} \right) \right) \]

\[+ \delta \frac{p}{p-1} \left( 1 - e^{(p-1)(t-T)} \right). \]
\[ e^{-t} \int_{t}^{T} e^{p(t-s)} V'(s) ds \]

\[ \overset{\text{(IBP)}}{=} \quad V(0) e^{(p-1)t} - e^{-t} V(t) + p e^{pt} \int_{t}^{T} e^{-ps} V(s) ds \]

\[ = \quad V(0) e^{(p-1)t} - e^{-t} V(t) + p e^{(p-1)t} \int_{t}^{T} e^{-(p-1)s} e^{-s} V(s) ds \]

\[ \leq \quad V(0) e^{(p-1)t} + e^{-t} V(t) \left( -1 + p e^{(p-1)t} \int_{t}^{T} e^{-(p-1)s} ds \right) \]

\[ + \delta p e^{(p-1)t} \int_{t}^{T} e^{-(p-1)s} dt \]

\[ = \quad V(0) e^{(p-1)t} + e^{-t} V(t) \left( -1 + \frac{p}{p-1} (1 - e^{(p-1)(t-T)}) \right) \]

\[ + \delta \frac{p}{p-1} (1 - e^{(p-1)(t-T)}). \]
\[ e^{-t} \int_{t}^{T} e^{p(t-s)} V'(s) ds \]

\[ \text{(IBP)} \quad V(0) e^{(p-1)t} - e^{-t} V(t) + pe^{pt} \int_{t}^{T} e^{-ps} V(s) ds \]

\[ = \quad V(0) e^{(p-1)t} - e^{-t} V(t) + pe^{(p-1)t} \int_{t}^{T} e^{-(p-1)s} e^{-s} V(s) ds \]

\[ \leq \quad V(0) e^{(p-1)t} + e^{-t} V(t) \left( -1 + pe^{(p-1)t} \int_{t}^{T} e^{-(p-1)s} ds \right) \]

\[ + \delta pe^{(p-1)t} \int_{t}^{T} e^{-(p-1)s} dt \]

\[ = \quad V(0) e^{(p-1)t} + e^{-t} V(t) \left( -1 + \frac{p}{p-1} (1 - e^{(p-1)(t-T)}) \right) \]

\[ + \delta \frac{p}{p-1} (1 - e^{(p-1)(t-T)}). \]
\[
\begin{align*}
&\quad e^{-t} \int_t^T e^{p(t-s)} V'(s) ds \\
&\text{(IBP)} \quad V(0) e^{(p-1)t} - e^{-t} V(t) + pe^{pt} \int_t^T e^{-ps} V(s) ds \\
&= V(0) e^{(p-1)t} - e^{-t} V(t) + pe^{(p-1)t} \int_t^T e^{-(p-1)s} e^{-s} V(s) ds \\
&\leq V(0) e^{(p-1)t} + e^{-t} V(t) \left(-1 + pe^{(p-1)t} \int_t^T e^{-(p-1)s} ds \right) \\
&\quad + \delta pe^{(p-1)t} \int_t^T e^{-(p-1)s} dt \\
&= V(0) e^{(p-1)t} + e^{-t} V(t) \left(-1 + \frac{p}{p-1} \left(1 - e^{(p-1)(t-T)} \right) \right) \\
&\quad + \delta \frac{p}{p-1} \left(1 - e^{(p-1)(t-T)} \right).
\end{align*}
\]
Therefore we see that
\[
\lim_{p \to \infty} \lim_{t \to -\infty} f_p(t) = 0.
\]
Thus if \( \hat{g} \) approximates \( \tilde{g} \) sufficiently well, then
\[
e^{-t} \| \hat{g} \|_t^2 \leq (1 + \varepsilon)^2 \pi (\text{diam}(X))^2 \| \tilde{g} \|^2 + \varepsilon
\]
for any positive \( \varepsilon \). Together with the estimate
\[
\| \xi_g \|_{t_*} \geq \left| \int_Z \hat{g} \tilde{g} e^{-\psi} \, dA \right| / \| \hat{g} \|_t \geq (1 - \varepsilon) \| \tilde{g} \|^2 / \| \hat{g} \|_t,
\]
we have
\[
\lim_{t \to -\infty} \| \xi_g \|_{t_*}^2 e^t \geq \frac{1}{\pi (\text{diam}(X))^2} \int_Z \| \tilde{g} \|^2 e^{-\psi} \, dA,
\]
as needed.

It follows now that
\[
C(t) := \sup_{g \in \mathcal{C}^\infty_0(Z)} \| \xi_g \|_{t_*}^{-2} \int_Z \| \tilde{g} \|^2 e^{-\psi} \, dA \leq \pi (\text{diam}(X)),
\]
and we have proved the Extension Theorem.
Therefore we see that

$$\lim_{p \to \infty} \lim_{t \to -\infty} f_p(t) = 0.$$ 

Thus if $\hat{g}$ approximates $\tilde{g}$ sufficiently well, then

$$e^{-t} \|\hat{g}\|_t^2 \leq (1 + \varepsilon)^2 \pi (\text{diam}(X))^2 \|\tilde{g}\|^2 + \varepsilon$$

for any positive $\varepsilon$. Together with the estimate

$$\|\xi g\|_{t^*} \geq \left| \int_Z \hat{g} \tilde{g} e^{-\psi} \, dA \right| / \|\hat{g}\|_t \geq (1 - \varepsilon) \|\tilde{g}\|^2 / \|\hat{g}\|_t,$$

we have

$$\lim_{t \to -\infty} \|\xi g\|_{t^*} e^t \geq \frac{1}{\pi (\text{diam}(X))^2} \int_Z |\tilde{g}|^2 e^{-\psi} \, dA,$$

as needed.

It follows now that

$$C(t) := \sup_{g \in \mathcal{C}_0^\infty(Z)} \|\xi g\|_{t^*}^{-2} \int_Z |\tilde{g}|^2 e^{-\psi} \, dA \leq \pi (\text{diam}(X)),$$

and we have proved the Extension Theorem.
Therefore we see that
\[ \lim_{p \to \infty} \lim_{t \to -\infty} f_p(t) = 0. \]
Thus if \( \hat{g} \) approximates \( \tilde{g} \) sufficiently well, then
\[ e^{-t} \| \hat{g} \|_t^2 \leq (1 + \varepsilon)^2 \pi (\text{diam}(X))^2 \| \tilde{g} \|^2 + \varepsilon \]
for any positive \( \varepsilon \). Together with the estimate
\[ \| \xi_g \|_{t^*} \geq \left| \int_Z \hat{g} \tilde{g} e^{-\psi} dA \right| / \| \hat{g} \|_t \geq (1 - \varepsilon) \| \tilde{g} \|^2 / \| \hat{g} \|_t, \]
we have
\[ \lim_{t \to -\infty} \| \xi_g \|_{t^*}^2 e^t \geq \frac{1}{\pi (\text{diam}(X))^2} \int_Z |\tilde{g}|^2 e^{-\psi} dA, \]
as needed.

It follows now that
\[ C(t) := \sup_{g \in C_0^\infty(Z)} \| \xi_g \|_{t^*}^{-2} \int_Z |\tilde{g}|^2 e^{-\psi} dA \leq \pi (\text{diam}(X)), \]
and we have proved the Extension Theorem.
Therefore we see that
\[
\lim_{p \to \infty} \lim_{t \to -\infty} f_p(t) = 0.
\]

Thus if \( \hat{g} \) approximates \( \tilde{g} \) sufficiently well, then
\[
e^{-t} \| \hat{g} \|_t^2 \leq (1 + \varepsilon)^2 \pi (\text{diam}(X))^2 \| \tilde{g} \| ^2 + \varepsilon
\]
for any positive \( \varepsilon \). Together with the estimate
\[
\| \xi_g \|_{t^*} \geq \left| \int_Z \hat{g} \tilde{g} e^{-\psi} dA \right| / \| \hat{g} \|_t \geq (1 - \varepsilon) \| \tilde{g} \|^2 / \| \hat{g} \|_t,
\]
we have
\[
\lim_{t \to -\infty} \| \xi_g \|^2_{t^*} e^t \geq \frac{1}{\pi (\text{diam}(X))^2} \int_Z |\tilde{g}|^2 e^{-\psi} dA,
\]
as needed.

It follows now that
\[
C(t) := \sup_{g \in C^\infty_0(Z)} \| \xi_g \|_{t^*}^{-2} \int_Z |\tilde{g}|^2 e^{-\psi} dA \leq \pi (\text{diam}(X)),
\]
and we have proved the Extension Theorem.
Therefore we see that
\[
\lim_{p \to \infty} \lim_{t \to -\infty} f_p(t) = 0.
\]

Thus if \( \hat{g} \) approximates \( \tilde{g} \) sufficiently well, then
\[
e^{-t} \| \hat{g} \|_{t}^2 \leq (1 + \varepsilon)^2 \pi (\text{diam}(X))^2 \| \tilde{g} \|^2 + \varepsilon
\]
for any positive \( \varepsilon \). Together with the estimate
\[
\| \xi_g \|_{t^*} \geq \left| \int_Z \hat{g} \tilde{g} e^{-\psi} dA \right| / \| \hat{g} \|_t \geq (1 - \varepsilon) \| \tilde{g} \|^2 / \| \hat{g} \|_t,
\]
we have
\[
\lim_{t \to -\infty} \| \xi_g \|^2_{t^*} e^t \geq \frac{1}{\pi (\text{diam}(X))^2} \int_Z |\tilde{g}|^2 e^{-\psi} dA,
\]
as needed.

It follows now that
\[
C(t) := \sup_{g \in \mathcal{C}_0^\infty(Z)} \| \xi_g \|_{t^*}^{-2} \int_Z |\tilde{g}|^2 e^{-\psi} dA \leq \pi (\text{diam}(X)),
\]
and we have proved the Extension Theorem.
Thanks for your patience.