MAT 542: COMPLEX ANALYSIS I
PROBLEM SET 4

(1) (a) Let \( N \geq 0 \) be an integer. Find all entire holomorphic functions \( f \) such that
\[
\int_{C} \frac{|f(z)|^2}{(1 + |z|^2)^{N+2}} \, dx \wedge dy < +\infty.
\]
(b) Find all entire holomorphic functions \( f \) such that
\[
\sup_{z \in \mathbb{C}} \left| \frac{\partial^6 f(z)}{\partial z^6} \right| \leq 4.
\]

(2) Show that if \( f : \mathbb{H} \to \mathbb{H} \) is holomorphic then
\[
\frac{|f'(z)|}{\text{Im } f(z)} \leq \frac{1}{\text{Im } z} \quad \text{and} \quad \frac{|f(z) - f(z_0)|}{|f'(z)|} \leq \frac{|z - z_0|}{|z - \bar{z}_0|}, \quad z, z_0 \in \mathbb{H}.
\]

(3) Give an example of a sequence of smooth functions \( f_j : \mathbb{C} \to \mathbb{C} \) that converges uniformly to a function \( f \) that is not smooth. Must \( f \) be continuous?

(4) Give an example of a normal family \( \mathcal{F} \subset \mathcal{O}(\Omega) \) on a domain \( \Omega \subset \mathbb{C} \) that is not closed, i.e., there is a convergent sequence \( \{f_j\} \subset \mathcal{F} \) whose limit is not in \( \mathcal{F} \).

(5) Show that there exists a smooth diffeomorphism from \( \Omega_1 := \mathbb{D} - \{0, 1/2\} \) to \( \Omega_2 := \mathbb{D} - \{1/2, 3/4\} \). Show that no such diffeomorphism can be holomorphic.

(6) Let \( \Omega_1, \Omega_2 \) be domains in \( \mathbb{C} \), and let \( f : \Omega_1 \to \Omega_2 \) be an injective holomorphic map. Fix \( a \in \Omega_1 \), and let \( g \in \mathcal{O}(\Omega_2 \setminus \{f(a)\}) \).
   (a) Show that the singularity of \( g \) at \( f(a) \) is the same as the singularity of \( g \circ f \in \mathcal{O}(\Omega_1 \setminus \{a\}) \) at \( a \). That is to say, show that \( g \) has a removable singularity (resp. pole of order \( m \), essential singularity) at \( f(a) \) if and only if \( g \circ f \) has a removable singularity (resp. pole of order \( m \), essential singularity) at \( a \).
   (b) What can you say if the map \( f \) is holomorphic but not necessarily injective?

(7) Let \( z_1, z_2 \in \mathbb{D} \) be 2 distinct points, let \( w_1, w_2 \in \mathbb{D} \) be 2 distinct points, and let \( f : \mathbb{D} - \{z_1, z_2\} \to \mathbb{D} - \{w_1, w_2\} \) be an injective holomorphic map. Show that
\[
\left| \frac{z_1 - z_2}{1 - \bar{z}_1 \bar{z}_2} \right| = \left| \frac{w_1 - w_2}{1 - \bar{w}_1 \bar{w}_2} \right|.
\]