Homework 10 (due 4/25)

MAT 342: Applied Complex Analysis

Read Sections 69–73 from Chapter 5 and Sections 74–75 from Chapter 6.

Problem 1.

(i) Differentiate the Maclaurin series of $\frac{1}{1-z}$ in order to find the Maclaurin series for $\frac{1}{(1-z)^2}$.

(ii) By substituting $z+1$ for z in the the Maclaurin series that you found in part (i), derive the Taylor series representation for the function $\frac{1}{z^2}$ in the disk $|z+1| < 1$.

(iii) By substituting $\frac{1}{1-z}$ for z in the Maclaurin series that you found in part (i), derive the Laurent series representation for the function $\frac{1}{z^2}$, centered at the point $z_0 = 1$. What is the annulus in which the representation holds?

Problem 2. Show that the function

$$f(z) = \begin{cases} \frac{1-\cos z}{z^2}, & z \neq 0 \\ \frac{1}{2}, & z = 0 \end{cases}$$

is entire. *(Hint: Find the power series representation of the function* $\frac{1-\cos z}{z^2}$ *with center at* 0 *and show that it converges for all* $z \in \mathbb{C}$.*)*

Problem 3.

(i) Show that the Taylor series expansion of the function $\frac{1}{z}$, with center at 1, is

$$\frac{1}{z} = \sum_{n=0}^{\infty} (-1)^n (z-1)^n,$$

for $|z-1| < 1$.

(ii) Explain why the function $\operatorname{Log} z$ is analytic in the disk $|z-1|<1$.

(iii) For each point z with $|z-1|<1$ consider the straight line segment C_z starting at 1 and ending at z. Evaluate

$$\int_{C_z} \frac{1}{z} dz.$$

(Hint: You do not need to do any computation. Note that $\operatorname{Log} z$ *is an antiderivative of* $1/z$ *in the disk* $|z-1|<1$*.)*

(iv) Integrate each term of the Taylor series along the contour C_z from part (i) in order to prove that

$$\operatorname{Log} z = \sum_{n=0}^{\infty} (-1)^n \frac{(z-1)^{n+1}}{n+1},$$

for $|z-1|<1$. This is the Taylor series representation of $\operatorname{Log} z$ in the disk $|z-1|<1$.

Problem 4. Using the Taylor series representation of $\operatorname{Log} z$ from the last part of Problem 3, show that the function

$$f(z) = \begin{cases} \frac{\operatorname{Log} z}{z-1}, & z \neq 0,\ z \neq 1, \text{ and } -\pi < \operatorname{Arg}(z) < \pi \\ 1, & z = 1 \end{cases}$$

is analytic in its domain.

Problem 5. Use multiplication of power series in order to find the Taylor series expansion up to z^4 of the function

$$\frac{e^z}{z^2+1},$$

with center at the origin. On what disk is the Taylor series convergent?

Problem 6. Use division of power series in order to find the first three non-zero terms of the Laurent series expansion of the function

$$\frac{1}{\cos z - 1},$$

with center at the origin. What is the annulus of convergence?