Schwarz lemma and boundary Schwarz lemma for pluriharmonic mappings

Jian-Feng Zhu (朱 剑峰)

Huaqiao University, Xiamen, China
flandy@hqu.edu.cn

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Outline of Report

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The classical boundary Schwarz lemma

Theorem A. Suppose $f : \mathbb{D} \to \mathbb{D}$ is a holomorphic self-mapping of the unit disk $\mathbb{D}$ satisfying $f(0) = 0$, and further, f is analytic at $z = 1$ with $f(1) = 1$. Then, the following two conclusions hold:

- $f'(1) \geq 1$.
- $f'(1) = 1$ if and only if $f(z) \equiv z$.

Theorem A has the following generalization.

Theorem B. Suppose $f : \mathbb{D} \to \mathbb{D}$ is a holomorphic mapping with $f(0) = 0$, and, further, f is analytic at $z = \alpha \in \mathbb{T}$ with $f(\alpha) = \beta \in \mathbb{T}$. Then, the following two conclusions hold:

- $\overline{\beta} f'(\alpha)\alpha \geq 1$.
- $\overline{\beta} f'(\alpha)\alpha = 1$ if and only if $f(z) \equiv e^{i\theta} z$, where $e^{i\theta} = \beta\alpha^{-1}$ and $\theta \in \mathbb{R}$.

Remark that, when $\alpha = \beta = 1$, Theorem B coincides with Theorem A.

Generalizations of boundary Schwarz lemma

Theorem C. (Carathéodory-Cartan-Kaup-Wu Theorem) Let Ω be a bounded domain in $\mathbb{C}^n$, and let f be a holomorphic self-mapping of Ω which fixes a point $p \in \Omega$. Then

- The eigenvalues of $J_f(p)$ all have modulus not exceeding 1;
- $|\det J_f(p)| \leq 1$;
- if $|\det J_f(p)| = 1$, then f is a biholomorphism of Ω.

H. Wu, *Normal families of holomorphic mappings*, Acta. Math. **119**, 1967, 193-233.

Recently, Liu et al. established a new type of boundary Schwarz lemma for holomorphic self-mappings of strongly pseudoconvex domain in $\mathbb{C}^n$.
T. Liu and X. Tang, *Schwarz lemma at the boundary of strongly pseudoconvex domain in* $\mathbb{C}^n$, Math. Ann. **366**, 2016, 655-666.

Schwarz lemma for harmonic mappings

Theorem 1. Suppose that w is a harmonic self-mapping of $\mathbb{D}$ satisfying $w(0) = 0$. Then we have the following inequality holds.

$$|w(z)| \leq \frac{4}{\pi} \arctan\left(|z| \frac{|z| + \frac{\pi}{4}\Lambda_w(0)}{1 + \frac{\pi}{4}\Lambda_w(0)|z|}\right) := M(z) \quad \text{for} \quad z \in \mathbb{D}. \tag{1}$$

We remark here that

$$\frac{4}{\pi} \arctan\left(|z| \frac{|z| + \frac{\pi}{4}\Lambda_w(0)}{1 + \frac{\pi}{4}\Lambda_w(0)|z|}\right) \leq \frac{4}{\pi} \arctan|z|,$$

holds for all $z \in \mathbb{D}$, since $\Lambda_w(0) \leq \frac{4}{\pi}$. Furthermore, the equality holds if $|z| = 1$.

Boundary Schwarz lemma for harmonic mappings

By using Theorem 1, we establish the following new-type of boundary Schwarz lemma for harmonic mappings.

Theorem 2. Suppose that w is a harmonic self-mapping of $\mathbb{D}$ satisfying $w(0) = 0$. If w is differentiable at $z = 1$ with $w(1) = 1$, then we have the following inequality holds.

$$\mathrm{Re}[w_x(1)] = \mathrm{Re}[w_z(1) + w_{\bar{z}}(1)] \geq \frac{4}{\pi} \frac{1}{1 + \frac{\pi}{4}\Lambda_w(0)}. \tag{2}$$

The above inequality is sharp.

Proof of Theorem 2.

Since w is differentiable at $z = 1$, we know that

$$w(z) = 1 + w_z(1)(z-1) + w_{\overline{z}}(1)(\overline{z}-1) + \circ(|z-1|).$$

By using Theorem 1, we have

$$2\mathrm{Re}\left[w_z(1)(1-z) + w_{\overline{z}}(1)(1-\overline{z})\right] \geq 1 - (M(z))^2 + \circ(|z-1|). \tag{3}$$

Take $z = r \in (0,1)$ and letting $r \to 1^-$, it follows from $M(1) = 1$ that

$$\begin{aligned} 2\mathrm{Re}\left[w_z(1) + w_{\overline{z}}(1)\right] &\geq \lim_{r\to 1^-} \frac{1-M(r)^2}{1-r} \\ &= \frac{4}{\pi}\frac{2}{1+\frac{\pi}{4}\Lambda_w(0)}. \end{aligned} \tag{4}$$

Therefore we have

$$\mathrm{Re}[w_z(1) + w_{\overline{z}}(1)] \geq \frac{4}{\pi}\frac{1}{1+\frac{\pi}{4}\Lambda_w(0)}$$

as required. □

Remarks

It is known that a harmonic mapping w of $\mathbb{D}$ has the representation $w = h + \bar{g}$, where h and g are holomorphic in $\mathbb{D}$.
We add the symbol "Re" in Theorem 2 because $w_x(1)$ may not be real. However, if in additional assuming $\varphi = h - g$ is holomorphic at $z = 1$, then

$$\mathrm{Im}[w_z(1)] = 0 = \mathrm{Im}[w_{\bar{z}}(1)],$$

and the symbol "Re" in (2) can be removed.

To check the sharpness of (2), consider the real harmonic mapping

$$w(z) = \frac{2}{\pi} \arctan \frac{2x}{1 - x^2 - y^2} : \mathbb{D} \to (-1, 1),$$

where $z = x + iy \in \mathbb{D}$. It is not difficult to check that w satisfies all the assumptions of Theorem 2. Moreover, elementary calculations show that $\Lambda_w(0) = \frac{4}{\pi}$ and $w_x(1) = \frac{2}{\pi}$.

K-quasiconformal mapping

We say that a function $f : D \to \mathbb{C}$ is *absolutely continuous on lines*, abbreviated as *ACL*, in a domain D if for every closed rectangle $\Gamma \subseteq D$ with sides parallel to x and y axes, respectively, f is absolutely continuous on a.e. horizontal line and a.e. vertical line in Γ. It is known that the partial derivatives of such functions always exist a.e. in D.

Definition. Let $K \geq 1$ be a constant. A homeomorphism $f : D \to \Omega$ between domains D and Ω in $\mathbb{C}$ is K*-quasiconformal*, briefly K-q.c. in the following, if

- f is ACL in D, and
- $|f_{\overline{z}}(z)| \leq k|f_z(z)|$ a.e. in D, where $k = \frac{K-1}{K+1}$.

Harmonic quasiconformal mappings are natural the generalization of conformal mappings. Recently many researchers have studied this active topic and obtained many interesting results.

Boundary Schwarz lemma for harmonic K-q.c.

For $L > 0$, $\Phi_L(s)$ is the Hersch-Pfluger distortion function defined by the equalities

$$\Phi_L(s) := \mu^{-1}(\mu(s)/L)\,, \quad 0 < s < 1; \quad \Phi_L(0) := 0\,, \; \Phi_L(1) := 1,$$

where $\mu(s)$ stands for the module of Grötzsch's extremal domain $\mathbb{D} \backslash [0, s]$. Let

$$L_K := \frac{2}{\pi} \int_0^{\frac{1}{\sqrt{2}}} \frac{d\left(\Phi_{1/K}(s)^2\right)}{s\sqrt{1-s^2}}. \tag{5}$$

Then L_K is a strictly decreasing function of K such that

$$\lim_{K\to 1} L_K = L_1 = 1 \quad \text{and} \quad \lim_{K\to\infty} L_K = 0. \tag{6}$$

Boundary Schwarz lemma for harmonic K-q.c.

Theorem 3. Let w be a harmonic K-quasiconformal self-mapping of $\mathbb{D}$. If w is differential at 1 with $w(0) = 0$ and $w(1) = 1$, then

$$\mathrm{Re}[w_x(1)] \geq M(K) := \max\left\{\frac{2}{\pi},\ L_K\right\}, \tag{7}$$

where L_K is given by (5). Furthermore, if $K = 1$, then (7) can be rewritten as follows

$$w_z(1) \geq 1 \tag{8}$$

which coincides with Theorem A.

General form

Theorem 4. Suppose that w is a harmonic self-mapping of $\mathbb{D}$ satisfying $w(a) = 0$. If w is differentiable at $z = \alpha$ with $w(\alpha) = \beta$, where $\alpha, \beta \in \mathbb{T}$, then we have the following inequality holds.

$$\mathrm{Re}\left\{\overline{\beta}\left[w_x(\alpha)\right]\right\} \geq \frac{4}{\pi}\frac{1}{1+\frac{\pi}{4}\Lambda_w(a)(1-|a|^2)}\frac{1-|a|^2}{|1-\bar{a}\alpha|^2}. \tag{9}$$

When $\alpha = \beta = 1$ and $a = 0$, then Theorem 4 coincides with Theorem 2.

Boundary Schwarz lemma for pluriharmonic mappings

For an $n \times n$ complex matrix A, we introduce the operator norm

$$\|A\| = \sup_{z \neq 0} \frac{\|Az\|}{\|z\|} = \max\{\|A\theta\| : \theta \in \partial\mathbb{B}^n\}. \quad (10)$$

Theorem 5. Let w be a pluriharmonic self-mapping of the unit ball $\mathbb{B}^n \subseteq \mathbb{C}^n$ satisfying $w(a) = 0$, where $a \in \mathbb{B}^n$. If $w(z)$ is differentiable at $z = \alpha \in \partial\mathbb{B}^n$ with $w(\alpha) = \beta \in \partial\mathbb{B}^n$, then we have the following inequality holds.

$$\mathrm{Re}\left\{\overline{\beta}^T \left[w_z(\alpha)\frac{1-\overline{a}^T\alpha}{1-|a|^2}(\alpha - a) + w_{\overline{z}}(\alpha)\frac{1-a^T\overline{\alpha}}{1-|a|^2}(\overline{\alpha}-\overline{a})\right]\right\} \tag{11}$$
$$\geq \frac{4}{\pi}\frac{1}{1+\frac{\pi}{4}\Lambda_w(a)\left|\frac{\alpha-a}{1-\overline{a}^T\alpha}\right|(1-|a|^2)},$$

where $\Lambda_w(a) = \|w_z(a)\| + \|w_{\overline{z}}(a)\|$. If $a = 0$, then we have

$$\mathrm{Re}\left\{\overline{\beta}^T[w_x(\alpha)]\right\} \geq \frac{4}{\pi}\frac{1}{1+\frac{\pi}{4}\Lambda_w(0)}. \tag{12}$$

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