On an extremal problem in geometric function theory

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Notation

- $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$: unit disk.
- H^2: the space of all $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in $\mathbb{D}$ such that

$$\|f\|_2 = \left(\sum_{n=0}^{\infty} |a_n|^2 \right)^{1/2} < \infty .$$

  On the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, each such function has radial limits a.e. $f(\zeta) = \lim_{r \to 1^-} f(r\zeta)$.
- H^∞: $\|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)| < \infty$.
  $\|f\|_2 \le \|f\|_\infty$.
- Can also define other H^p spaces (not needed here).

Linear Extremal Problems

- Let $1 \leq p \leq \infty$. Find

$$\sup \left\{ |\Phi(a_0, \dots, a_n)| : \; \|f\|_{H^p} \leq 1 \, , f(z) = \sum_{j=0}^{\infty} a_j z^j \right\}$$

  or

$$\sup \left\{ \operatorname{Re} \Phi(a_0, \dots, a_n) : \; \|f\|_{H^p} \leq 1 \, , f(z) = \sum_{j=0}^{\infty} a_j z^j \right\}$$

  where Φ is a continuous functional.
- Carathéodory, Fejér, Kakeya, Landau, Pick, Schur, Szász, Szegö (1910–1930): concrete problems of this type and related interpolation questions.
- Unified approach, dual extremal problems (around 1950): Rogosinski- H. Shapiro and S.Ya. Khavinson.

A non-linear extremal problem

- Let $\mathcal{B}^*$ denote the class of all f analytic in $\mathbb{D}$ such that $0 < |f(z)| \leq 1$ for all z en $\mathbb{D}$.
- Consider the extremal problem of determining the supremum:

$$M_n = \sup \{|a_n| : \ f \in \mathcal{B}^*\} , \quad n \geq 1 .$$

- Obviously, $M_n \leq 1$ for all $n \geq 1$. Uniform bounds on M_n smaller than one exist!
- Standard arguments (normal families, Hurwitz's theorem) show that the supremum M_n is attained for some function f (called an *extremal function*).

Some basic observations

- Clearly, if $f(z) = \sum_{j=0}^{\infty} a_j z^j$ is extremal and $|\lambda| = |\mu| = 1$, then so is $\lambda f(\mu z) = \lambda \sum_{j=0}^{\infty} \mu^j a_j z^j$ (rotations of f).
- Thus, we can study instead the equivalent "normalized" problem of finding

  $$M_n = \sup\{\mathrm{Re}\, a_n : \ f \in \mathcal{B}_* \, , \ a_0 > 0\} \, ,$$

  for $n \geq 1$.
- The n-th coefficient of an extremal function for the above problem must actually satisfy the condition

  $$\mathrm{Re}\, a_n = |a_n| \, .$$

  That is, $a_n > 0$ actually must hold for such functions. (Otherwise get a function in $\mathcal{B}_*$ for which $\mathrm{Re}\, a_n > M_n$.)

The Krzyż conjecture

- For every $n \geq 1$, the function

$$f_n(z) = e^{(z^n-1)/(z^n+1)} = \frac{1}{e} + \frac{2}{e}z^n - \frac{2}{3e}z^{3n} + \dots$$

  shows that $M_n \geq 2/e = 0.73575888\dots$.
- **Conjecture.** (Jan Krzyż, 1968) $M_n = 2/e$ for all $n \geq 1$.

  Moreover, for any fixed $n \geq 1$, equality is attained only for the function f_n as above:

$$f_n(z) = e^{(z^n-1)/(z^n+1)} = \frac{1}{e} + \frac{2}{e}z^n - \frac{2}{3e}z^{3n} + \dots$$

  (and its rotations if we maximize the modulus of $|a_n|$).

Why is this problem of interest?

- It is quite a natural problem and simple to formulate.
- Being a non-linear extremal problem, it is a lot harder than most linear ones.
- There is some remote analogy with the Bieberbach conjecture in the theory of univalent functions.
- The problem can be studied from many points of view and is related to many different concepts and topics, e.g., the Toeplitz-Carathéodory criterion for functions with positive real part, positive semi-definite quadratic forms and matrices, Herglotz representation, Loewner chains, the universal analytic covering map of the punctured disk, inner functions and Hardy space theory, polynomials with positive real part on the circle, the Bateman function, Laguerre polynomials, etc.

Some known uniform estimates

It is not at all easy to show that $M_n \leq C < 1$ for a uniform constant M (not depending on n).

- Horowitz (1978):

$$M_n \leq 1 - \frac{1}{3\pi} + \frac{4}{\pi} \sin \frac{1}{12} = 0.99987\ldots, n \in \mathbb{N}.$$

- This was improved to 0.9991 . . . in Ermers' thesis (1990).
- Asymptotic bounds: Prokhorov, Romanova (2006).

Partial results

- Case $n = 1$: simple (known since 1934).
- $n = 2$ is already non-trivial: Krzyż, Reade, MacGregor (unpublished).
- $n = 3$: Hummel, Scheinberg, Zalcman (1977).
- $n = 4$: Tan (1983), Brown (1987).
- $n = 5$: Samaris (2003).
- Lewandowski, Szynal, Brown, Koepf, Schmersau, Peretz (among others): various partial contributions.

Properties of extremal functions

- Hummel, Scheinberg, and Zalcman (1977) established the structure of extremal functions.
- This structure can actually be deduced from an earlier work of S.Ya. Khavinson.
- Other proofs: Kortram (1993), based on the study of extreme points of the unit ball in H^∞.
- Martín, Sawyer, Uriarte-Tuero, and Vukotić (2015): more information on the properties of extremal functions.
- Agler, McCarthy (2018): relations to an entropy conjecture.

Some algebra. Recurrence relations

Lemma. If f and g are analytic in D and

$$f = e^g\,, \quad f(z) = \sum_{j=0}^{\infty} a_j z^j\,, \quad g(z) = \sum_{j=0}^{\infty} b_j z^j\,,$$

then

$$a_0 = e^{b_0}$$

and

$$a_n = \sum_{k=0}^{n-1} \left(1 - \frac{k}{n}\right) a_k b_{n-k}\,,\ n \geq 1\,.$$

More precisely,

$$\begin{aligned} a_1 &= a_0 b_1\,, \quad a_2 = a_0 \left(\frac{b_1^2}{2} + b_2 \right)\,, \\ a_3 &= a_0 \left(\frac{b_1^3}{6} + b_1 b_2 + b_3 \right)\,, \\ a_4 &= a_0 \left(\frac{b_1^4}{24} + \frac{b_1^2 b_2}{2} + b_1 b_3 + \frac{b_2^2}{2} + b_4 \right)\,, \dots \end{aligned}$$

There is an explicit general formula, due to Z. Lewandowski and J. Szynal.

In general:

$$a_n = a_0\, P_n(b_1, b_2, \ldots, b_n)\,,$$

where, for each $n \geq 1$,

$$P_n(b_1, b_2, \ldots, b_n) = \sum_{\substack{1 \leq m \leq n,\\ \sum_{j=1}^{m} i_j(n) = n}} c_{i_1(n), i_2(n), \ldots, i_m(n)}\, b_{i_1(n)} b_{i_2(n)} \ldots b_{i_m(n)}\,,$$

where all $i_j(n) \in \mathbb{N}$ and

$$c_{i_1(n), i_2(n), \ldots, i_m(n)} > 0\,.$$

These coefficients can be computed explicitly.

Some observations

- No more than one coefficient of f can be $\geq 2/e$ since if $|a_n| \geq 2/e$ and $|a_m| \geq 2/e$, $m \neq n$, then

$$1 \geq \|f\|_\infty^2 \geq \|f\|_2^2 = \sum_{n=0}^{\infty} |a_n|^2 \geq 2\left(\frac{2}{e}\right)^2 = \frac{8}{e^2} > 1\,.$$

- For the purported extremal function, we have

$$a_0 = \frac{1}{e}\,, \quad a_1 = a_2 = \ldots = a_{n-1} = 0\,, \quad a_n = 2a_0\,.$$

Proving just one of these relations for an extremal function could constitute fundamental progress.

The case $n = 1$

If $f \in \mathcal{B}_*$ then we can write $f = e^g$ where g is a function analytic in $\mathbb{D}$ and with negative real part, hence $g = (h+1)/(h-1)$, for some h with $\|h\|_\infty \le 1$. A direct computation shows that

$$f' = -\frac{2h'}{(h-1)^2} e^{\dfrac{h+1}{h-1}} \,,$$

hence by the Schwarz-Pick lemma

$$|f'(0)| = \frac{2|h'(0)|}{|1-h(0)|^2}\, e^{\mathrm{Re}\,\frac{h(0)+1}{h(0)-1}} \le 2\frac{1-|h(0)|^2}{|1-h(0)|^2}\, e^{-\dfrac{1-|h(0)|^2}{|1-h(0)|^2}} \le \frac{2}{e}$$

since the function $u(x) = 2x\, e^{-x}$ considered in $[0, +\infty)$ attains its maximum at the point $x = 1$.
The case of equality requires some analysis.

Inequalities of Carathéodory and Livingston

Class $\mathcal{P}$: analytic functions in $\mathbb{D}$ with $\operatorname{Re} f(z) > 0$ and $f(0) = 1$.

Lemma. (Carathéodory) If $f \in \mathcal{P}$, $f(z) = 1 + \sum_{n=1} p_n z^n$ then$|p_n| \leq 2$ for all $n \geq 1$. Equality holds if and only if the representing Herglotz measure is supported on some subsets of the n-th roots of unity.

Theorem. (Livingston 1969; generalization: I. Efraimidis, 2016). Under the same assumptions,

$$|p_n - w p_k p_{n-k}| \leq 2 \max\{1, |1 - 2w|\} .$$

The case $n = 2$: one possible proof

$$a_2 = a_0 \left(\frac{b_1^2}{2} + b_2 \right) .$$

Write

$$g(z) = b_0 + b_1 z + b_2 z^2 + \ldots = b_0(1 + p_1 z + p_2 z^2 + \ldots)$$

where the function between parentheses belongs to the Carathéodry class. Then

$$|a_2| = e^{-|b_0|} |b_0| \left| p_2 - \left(-\frac{b_0}{2} p_1^2 \right) \right| \le \frac{2}{e}$$

by the previous inequalities ($n = 2$, $k = 1$, $w = -b_0/2$) in the case when $-2 < b_0 < 0$. The case $b_0 \ge -2$ can also be handled in a simple way.

The structure of extremal functions

- **Theorem.** An extremal function for M_n must have the form

$$f(z) = e^{\sum_{j=1}^{N} r_j \frac{\alpha_j z - 1}{\alpha_j z + 1}},$$

$$1 \le N \le n, \ r_j > 0, \ |\alpha_j| = 1, \ j = 1, 2, \dots, N.$$

One possible proof (D. Marshall): open mapping theorem.

- There are many parameters to handle and it is not at all obvious that we must have a complete symmetry:

$$N = n, \quad r_j = \frac{1}{n}, \quad \alpha_j = e^{2\pi j i/n}.$$

- Note that the above f is actually $f = e^{\frac{B-1}{B+1}}$, where B is a finite Blaschke product of degree $N \le n$. Need to show: $B(z) = z^n$.

Some relevant observations

If $f(z) = \sum_{j=0}^{\infty} a_j z^j$ is extremal for the Krzyż problem (with $a_0 > 0$) then we know that $a_n > 0$.
Also, one can use a variational method to show that

$$P(\lambda) = a_n + 2a_{n-1}\lambda + \ldots + 2a_1\lambda^{n-1} + 2a_0\lambda^n$$

has positive real part on the unit circle (equivalently, on the closed unit disk).
Consequence: $a_n \geq 2a_0$.

Can also show: if f is extremal, $f = e^g$, and the coefficients of g are denoted by b_j, we have

$$\mathrm{Re}\,\{a_n b_0 + a_{n-1} b_1 + \ldots + a_0 b_n\} = 0\,.$$

Writing

$$P(z) = 2a_0 \prod_{k=1}^{n} (z - \lambda_k)\,,$$

and computing $P(0)$, it follows that the zeros λ_k of the polynomial $P(\lambda)$, $1 \le k \le n$, satisfy the condition

$$(-1)^n \prod_{k=1}^{n} \lambda_k \ge 1\,.$$

Some special results

One can prove the Krzyż conjecture under some additional hypotheses on an extremal function such as: $a_i = 0$ for all i belonging to some $I \subset \{1, 2, \ldots, n-1\}$. Typically, "about a half of these initial coefficients" are assumed to vanish.

J.E. Brown's (1985): if $a_i = 0$ whenever $1 \le i < (n+1)/2$ then the conjecture follows.

Peretz (1991) proved the following:
(a) if n is odd and $a_1 = a_3 = \ldots = a_{n-2} = 0$ then $a_0 \le 1/e$,
(b) if, in addition to this, $a_0 = 1/e$ then $a_n = 2/e$.

Note: Brown's assumptions

$$a_i = 0 \quad \text{whenever} \quad 1 \le i < (n+1)/2$$

imply that also $b_i = 0$ whenever $1 \le i < (n+1)/2$.

Also simple: for n odd, Peretz's assumptions

$$a_1 = a_3 = \ldots = a_{n-2} = 0$$

imply that $a_k = a_0 b_k$ for each $k \in \{1, 2, \ldots, n\}$, hence also

$$b_1 = b_3 = \ldots = b_{n-2} = 0\,.$$

Inductive sets

In what follows, the sets I of indices as in the papers by Brown and Peretz will be called inductive sets. This will lead to further examples and a unified proof of the conjecture under other similar assumptions.

Fix $n \in \mathbb{N}$, $n \geq 2$. Given $K \subset \{1, 2, 3, ..., n-1\}$, define

$$\mathbb{C}^n_K = \{c = (c_1, c_2, ..., c_n) \in \mathbb{C}^n : \; c_i = 0 \text{ for all } i \in K\} .$$

By an *additive semigroup* or simply *semigroup* we will mean a subset of $\mathbb{N}$ closed under addition. For $K \subset \{1, 2, 3, ..., n-1\}$, denote by $G(K)$ the additive semigroup generated by $(K \cup \{n\})^c = \mathbb{N} \setminus (K \cup \{n\})$.

Let $n \in \mathbb{N}$, $n \geq 2$, and $a_0 > 0$.
A subset I of $\{1, 2, 3, ..., n-1\}$ is said to be *n-inductive* if $a_n = a_0 b_n$ for *all* $a \in \mathbb{C}_I^n$ and $b \in \mathbb{C}^n$ that satisfy the recursion formulas for a_j and b_j.
A subset J of $\{1, 2, 3, ..., n-1\}$ is said to be *exponentially n-inductive* if $a_n = a_0 b_n$ for *all* $a \in \mathbb{C}^n$ and $b \in \mathbb{C}_J^n$ that satisfy the recursion formulas for a_j and b_j.

We will sometimes suppress the integer n and simply say I is *inductive* or J is *exponentially inductive* when the value of n is understood.

The following lemma allows us to have enough examples.

Lemma. Fix $n \in \mathbb{N}$, $n \geq 2$, $a_0 > 0$.
(a) Let $I = \{i : 1 \leq i \leq n-1, a_i = 0\}$. Then I is n-inductive if and only if $n \notin G(I)$.
(b) Let $J = \{j : 1 \leq j \leq n-1, b_j = 0\}$. Then J is exponentially n-inductive if and only if $n \notin G(J)$.

The situation considered by Peretz corresponds to the semigroup $G(I) = 2\mathbb{N} = \{2, 4, 6, 8, \ldots\}$.

In Brown's result, $G(I)$ is the semigroup generated by the set $\{i \in \mathbb{N} : \ (n+1)/2 \le i \le n-1\}$.

In both cases, as observed before, $I = J$ hence $G(I) = G(J)$.

There are other examples of inductive sets with density about $1/2$ in $1, 2, \ldots, n-1$.

Summary of observations

Let $f = e^g$ be extremal and let a_j and b_j be as before. Then $a_n > 0$ (in fact, $a_n = M_n \geq 2/e$). May assume: $b_0 < 0$. Know:

(I) $2a_0 \leq a_n$ (variation).

(II) $N \leq n$ in the formula for the structure of extremal functions

$$f(z) = e^{\sum_{j=1}^{N} r_j \frac{\alpha_j z - 1}{\alpha_j z + 1}},$$

and also $\operatorname{Re} b_n \leq 2|b_0|$ (by Carathéodory's lemma).

(III) The polynomial $P(z) = a_n + 2a_{n-1}z + \ldots 2a_1 z^{n-1} + 2a_0 z^n$ has non-negative real part on the closed unit disk $\overline{\mathbb{D}}$ and strictly positive real part on $\mathbb{D}$.

(IV) The zeros λ_j, $1 \leq j \leq n$, of the polynomial P satisfy $|\lambda_k| \geq 1$ for $1 \leq k \leq n$ and $(-1)^n \prod_{k=1}^{n} \lambda_k \geq 1$.

A trick of F. Wiener

If $|f| \leq 1$ in $\mathbb{D}$, $f(z) = \sum_{k=0}^{\infty} a_k z^k$, and $\omega = e^{2\pi i/n}$ is the primitive n-th root of unity, define the function

$$W_n f(z) = \frac{1}{n} \sum_{k=0}^{n-1} f(\omega^k z) = \sum_{k=0}^{\infty} a_{nk} z^{nk} = H(z^n)\,.$$

$$H(z) = \sum_{k=0}^{\infty} a_{nk} z^k = a_0 + a_n z + a_{2n} z^2 + \dots\,,$$

also bounded by one and with $H'(0) = a_n$.

Various equivalent forms of the conjecture

Theorem. Let $n \geq 2$ and consider an arbitrary but fixed extremal function f for the normalized Krzyż problem. Write $f(z) = \sum_{j=0}^{\infty} a_j z^j$, $g(z) = \sum_{j=0}^{\infty} b_j z^j$, and $f = e^g$ as before, and consider the quantity B_0, the polynomial P and its zeros as described.

(I) The following statements are equivalent:

(1) $a_n = 2a_0$;
(2) $a_k = 0$ whenever $1 \leq k < n$;
(3) $b_k = 0$ whenever $1 \leq k < n$;

(4) $f(z) = e^{(z^n-1)/(z^n+1)}$ (and, in particular, $M_n = 2/e$);
(5) the set I consisting of all indices $i \in \{1, 2, \ldots, n-1\}$ for which $a_i = 0$ is n-inductive;
(6) the set J consisting of all indices $j \in \{1, 2, \ldots, n-1\}$ for which $b_j = 0$ is exponentially n-inductive;
(7) there exists an analytic function H in $\mathbb{D}$ such that $|H(z)| \leq 1$ for all $z \in \mathbb{D}$ and $k \in \mathbb{N}$, $k \geq n/2$, with

$$g(z) = (z^k H(z) - 1)/(z^k H(z) + 1)\,;$$

(8) the zeros of the polynomial P satisfy $(-1)^n \prod_{k=1}^n \lambda_k = 1$;
(9) the zeros of P all lie on the unit circle;
(10) the zeros of P are actually the n-th roots of -1;

(11) $a_n b_0 + a_0 \mathrm{Re}\{b_n\} = 0$;
(12) $\mathrm{Re}\{a_1 b_{n-1} + a_2 b_{n-2} + \ldots + a_{n-1} b_1\} = 0$;
(13) $a_n = a_0 \,\mathrm{Re}\, b_n$.
(14) Re $b_n = 2|b_0|$, $N = n$, and $r_1 = r_2 = \ldots = r_n$;
(15) $B_0 = 1$; $W_n f$ does not vanish in $\mathbb{D}$;
(16) $W_n f \equiv f$;
(17) $g = W_n g$.

(II) In addition to the above, the following is true: there is a unique extremal function for the normalized problem if and only if every extremal function for this problem satisfies any one of the conditions (1)–(17) from part (I), and therefore all of them.

(1) $\Rightarrow$ (2): Know that the coefficients of any extremal function f satisfy:

$$\mathrm{Re}\,\{a_n + 2a_{n-1}\lambda + \ldots + 2a_1\lambda^{n-1} + 2a_0\lambda^n\} \geq 0\,.$$

whenever $|\lambda| = 1$.
Write

$$\omega_k = e^{(2k+1)\pi i/n}\,, \quad k = 0, 1, \ldots, n-1\,,$$

for the n-th roots of -1. By the assumption $\mathrm{Re}\,a_n = 2a_0$, for each value

$$\lambda = \omega_k, \quad k = 0, 1, \ldots, n-1\,,$$

we get

$$\mathrm{Re}\,\{a_n + 2\omega_k^n a_0\} = 0\,.$$

Thus, for each $k = 0, 1, \ldots, n-1$ we get

$$\mathrm{Re}\,\{a_{n-1}\omega_k + \ldots + a_1\omega_k^{n-1}\} \geq 0 \tag{1}$$

By basic algebra, for any fixed j with $1 \leq j \leq n-1$ we have

$$\sum_{j=0}^{n-1} \omega_k^j = 0\,.$$

Summing up over $k = 0, 1, \ldots, n-1$ all terms that appear on the left in (1), we get

$$\sum_{k=0}^{n-1} \mathrm{Re}\,\{a_{n-1}\omega_k + \ldots + a_1\omega_k^{n-1}\} = 0\,.$$

Since every summand is non-negative by (1), each one of them has to be zero:

$$\mathrm{Re}\,\{a_{n-1}\omega_k + \ldots + a_1\omega_k^{n-1}\} = 0\,,$$

$k = 0, 1, \ldots, n-1$, hence also

$$\mathrm{Re}\,\{a_n + 2a_{n-1}\omega_k + \ldots + 2a_1\omega_k^{n-1} + 2a_0\omega_k^n\} = 0\,,$$

$k = 0, 1, \ldots, n-1$.

When $\lambda = e^{it}$, $t \in [0, 2\pi]$, the restriction of Re P to the unit circle:

$$T(t) = \mathrm{Re}\,\{a_n + 2a_{n-1}\lambda + \ldots + 2a_1\lambda^{n-1} + 2a_0\lambda^n\}$$

is a trigonometric polynomial of t of degree n and with real coefficients. Since $T(t) \geq 0$ on the circle, Fejér's lemma tells us that

$$T(t) = |(\gamma_0 + \gamma_1\lambda + \ldots \gamma_n\lambda^n)^2|\,, \quad \lambda = e^{it}\,.$$

The complex polynomial

$$P(z) = (\gamma_0 + \gamma_1 z + \dots \gamma_n z^n)^2$$

has $2n$ zeros counting the multiplicities, each zero being obviously of order at least two.
We know that this polynomial has at least n distinct zeros

$$\omega_k\,, \quad k = 0, 1, \dots, n-1\,,$$

which are roots of -1, so each one of these zeros must be double and hence P cannot have any other zeros.

Thus, the polynomial factorizes as

$$\begin{aligned} P(z) &= (\gamma_0 + \gamma_1 z + \ldots \gamma_n z^n)^2 \\ &= C \prod_{k=0}^{n-1} (z - \omega_k)^2 = C(z^n + 1)^2 \,. \end{aligned}$$

Hence

$$\mathrm{Re}\,\{a_n + 2a_{n-1}\lambda + \ldots + 2a_1\lambda^{n-1} + 2a_0\lambda^n\}$$

$$= |C(\lambda^n + 1)^2| = 2|C|\mathrm{Re}\,\{\lambda^n + 1\}$$

for all λ on the unit circle.

Simple exercise: two polynomials whose real parts are equal on the unit circle must coincide everywhere, except for an imaginary constant.
Thus, we have

$$a_n + 2a_{n-1}z + \ldots + 2a_1 z^{n-1} + 2a_0 z^n =$$

$$|C|(z^n + 1) + ic\,, \qquad z \in \mathbb{C}\,,\ c \in \mathbb{R}\,.$$

We know that actually $a_n > 0$, hence

$$c = 0\,,$$

$$a_n = 2a_0 = 2|C|\,,$$

$$a_1 = a_2 = \ldots = a_{n-1} = 0\,,$$

which proves (2).

<u>(2) $\Rightarrow$ (3):</u> Induction.

<u>(3) $\Rightarrow$ (4):</u> The recurrence relations

$$a_n = \sum_{k=0}^{n-1} \left(1 - \frac{k}{n}\right) a_k b_{n-k}\,, \;\; n \geq 1\,.$$

yield

$$a_n = a_0 b_n = |a_0 b_n| = \left|\frac{b_n}{b_0}\right| |b_0| e^{-|b_0|}\,.$$

The function g/b_0 belongs to the normalized class P so Carathéodory's lemma applies: $|b_n/b_0| \leq 2$. The function xe^{-x} achieves its maximum $1/e$ at $x = 1$. Thus, $|a_n| \leq 2/e$ and we must have $|a_n| = 2/e$.

An analysis of the case of equality yields that $b_0 = -1$, $b_n = 2$, hence f is the function we expected.

- According to the above findings, proving the conjecture essentially amounts to showing that $N = n$ in

$$f(z) = e^{\sum_{j=1}^{N} r_j \frac{\alpha_j z - 1}{\alpha_j z + 1}},$$

and the following three sets of numbers actually coincide:
- $\{\alpha_1, \ldots, \alpha_N\}$, the rotation coefficients in the point masses in extremal functions,
- $\{\omega_1, \ldots, \omega_n\}$, the n-th roots of -1,
- $\{\lambda_1, \ldots, \lambda_n\}$, the roots of the polynomial P associated with the extremal function f,
- the zeros of $\operatorname{Re} P = Q^2$ on $\mathbb{T}$.

- Alternatively, it suffices to show the uniqueness of the extremal function for the normalized problem. Although unpublished so far, this fact seems to be known to several experts.

Some new equivalent forms

Theorem. Let f be a singular inner function with N atoms, $N \leq n$. Then the following statements are equivalent:

(A) The Krzyż conjecture is true: $M_n = 2/e$ and, moreover, the only extremal function is $f(z) = e^{\frac{z^n-1}{z^n+1}}$.

(B) $N = n$, $b_n = 2$, and $b_0 = -1$.

(C) $N = n$ and $\operatorname{Re} b_n = 2|b_0|$.

(D) $N = n$, the set $\{\alpha_1, \ldots, \alpha_n\}$ coincides with the set of all n-th roots of some number α of modulus one and, moreover, $\operatorname{Re} P(\alpha_k) = 0$ for each $k \in \{1, 2, \ldots, n\}$.

(E) $N = n$ and the set $\{\alpha_1, \ldots, \alpha_n\}$ coincides with the set of all n-th roots of some number α of modulus one.

Note that this allows us to remove one requirement from an earlier condition

(14) $\operatorname{Re} b_n = 2|b_0|$, $N = n$, and $r_1 = r_2 = \ldots = r_n$.

THANK YOU!

$E\upsilon\chi\alpha\rho\iota\sigma\tau\omega$