Quasiconformal variation of cross-ratios and applications

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Variation of torus and cross-ratio under qc-mapping

$$\mathbb{H} \ni \tau = \text{modulus of torus} \xrightarrow{\text{elliptic modular function } \lambda(\tau)} \text{cross-ratio} \in \mathbb{C} \setminus \{0,1\}$$

1. Integral estimate on $\left|\log \dfrac{f(z_1)}{z_1} - \log \dfrac{f(z_2)}{z_2}\right| = \left|\log \dfrac{f(z_1)}{f(z_2)} - \log \dfrac{z_1}{z_2}\right|$

$$\frac{z_1}{z_2} = \frac{z_1 - 0}{z_2 - 0} \cdot \frac{z_2 - \infty}{z_1 - \infty}$$

- Conformality at a pt: Teichmüller-Wittich-Belinskiĭ's theorem
- $C^{1+\alpha}$-conformality: $f(z) = f(0) + f'(0)z + O(|z|^{1+\alpha})$
- Construction an entire function of class $\mathcal{B}$ with finite order having a wandering domain. D. Martí Pete's talk.

2. Differentiability of qc-mappings w.r.t. a parameter (Ahlfors-Bers)

- More precise estimate on the variation of torus
- An integral-differential equation for elliptic modular function $\lambda(\tau)$

Quasiconformal mappings

Definition: A mapping $f : \Omega \to \Omega'$ between open sets Ω, Ω' in $\mathbb{C}$ is called *quasiconformal* if

(i) f is an orientation-preserving homeomorphism;

(ii) f_x, f_y (in the sense of distribution) $\in L^1_{loc}(\mathbb{C})$;

(iii) Denote $\mu_f(z) = \dfrac{f_{\overline{z}}(z)}{f_z(z)}$ and $D_f(z) = \dfrac{1 + |\mu_f(z)|}{1 - |\mu_f(z)|}$, then

$$||\mu_f||_\infty < 1 \quad \text{(or equivalently } K(f) = \operatorname{ess\,sup} D_f(z) < \infty).$$

- Qc-mappings are almost everywhere differentiable but not everywhere.

- Quasi-invariance of conformal invariants– moduli of annuli, quadrilaterals.

- Measurable Riemann Mapping Theorem: Given $\mu \in L^\infty(\mathbb{C})$, $\exists\ f_\mu : \mathbb{C} \to \mathbb{C}$ quasiconformal with $\mu_{f_\mu} = \mu$ a.e. (unique with $f(0) = 0,\ f(1) = 1$)

- If $\mu = \mu_s \in L^\infty(\mathbb{C})$ is differentiable in s in L^∞ sense, then $f_{\mu_s}(z)$ is differentiable with respect to s.

Key Inequality. Fix $K \geq 1$. There exist $0 < \delta_1 < 1$ and $C > 0$ such that if $f : \mathbb{C} \to \mathbb{C}$ K-quasiconfromal , $f(0) = 0$ and $0 < |z_2| \leq \delta_1 |z_1|$, then

$$\left| \log \frac{f(z_1)}{z_1} - \log \frac{f(z_2)}{z_2} \right|_{\mathbb{C}/2\pi i\mathbb{Z}} \leq CJ(\mu_f; z_1, z_2),$$

where $\varphi_{z_1,z_2}(z) = \frac{z_1}{z(z-z_1)(z-z_2)}$ and

$$J(\mu, z_1, z_2) = 2 \left| \iint_{\mathbb{C}} \frac{\mu}{1-|\mu|^2} \varphi_{z_1,z_2}(z) dxdy \right| + 2 \iint_{\mathbb{C}} \frac{|\mu|^2}{1-|\mu|^2} |\varphi_{z_1,z_2}(z)| dxdy.$$

If $|z_2| \ll |z| \ll |z_1|$, then $\varphi_{z_1,z_2}(z) \sim -\frac{1}{z^2}$ (with integral estimates).

Theorem (Teichmüller, Wittich, Belinskiĭ) Let $f : \mathbb{C} \to \mathbb{C}$ qusasiconformal mapping, and assume that

$$\iint_{0<|z|<1} \frac{|\mu_f(z)| dx\, dy}{|z|^2} < \infty, \qquad (1)$$

classical proofs radial and angular differentiability separately

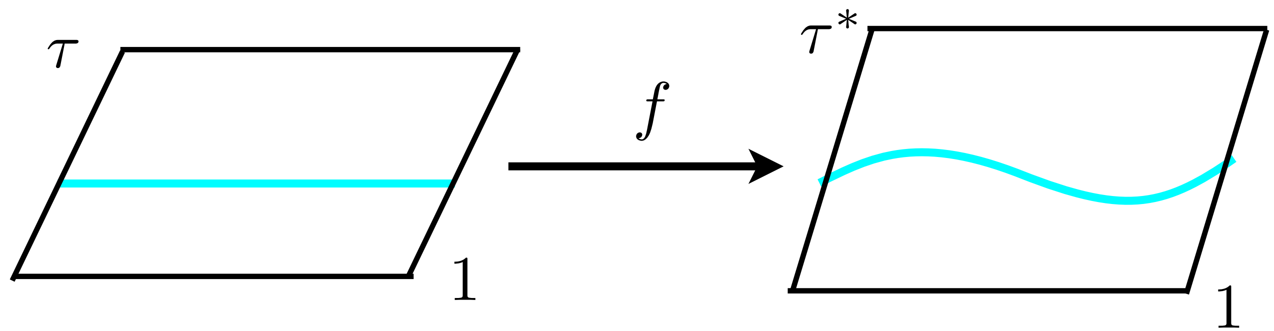
Then f is conformal at $z = 0$, i.e. $\exists \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} =: f'(0) \neq 0.$

Theorem (Gutlyanskiĭ-Martio) Same conclusion under weaker conditions:

$$\iint_{|z|<1} \frac{|\mu_f(z)|^2}{1-|\mu_f(z)|^2} \frac{dx\, dy}{|z|^2} < \infty \text{ and } \lim_{r \searrow 0} \iint_{r<|z|<1} \frac{\mu_f(z)}{1-|\mu_f(z)|^2} \frac{dx\, dy}{z^2} \text{ exists.}$$

The Grötzsch inequality (where it all starts)

$$E_\tau = \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau) \qquad\qquad E_{\tau^*} = \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau^*)$$



$$1 \le \int_0^1 |f_x|dx = \int_0^1 |f_z + f_{\overline{z}}|dx = \int_0^1 |1+\mu||f_z|dx$$

$$(\operatorname{Im}\tau)^2 = \left(\int_0^{\operatorname{Im}\tau} dy\right)^2 \le \left(\iint_{E_\tau} |1+\mu||f_z|dxdy\right)^2$$

$$\le \iint_{E_\tau} \underbrace{(1-|\mu|^2)|f_z|^2}_{\text{Jacobian } f} dxdy \iint_{E_\tau} \frac{|1+\mu|^2}{1-|\mu|^2} dxdy$$

$$= \operatorname{Im}\tau^* \iint_{E_\tau} \frac{|1+\mu|^2}{1-|\mu|^2} dxdy$$

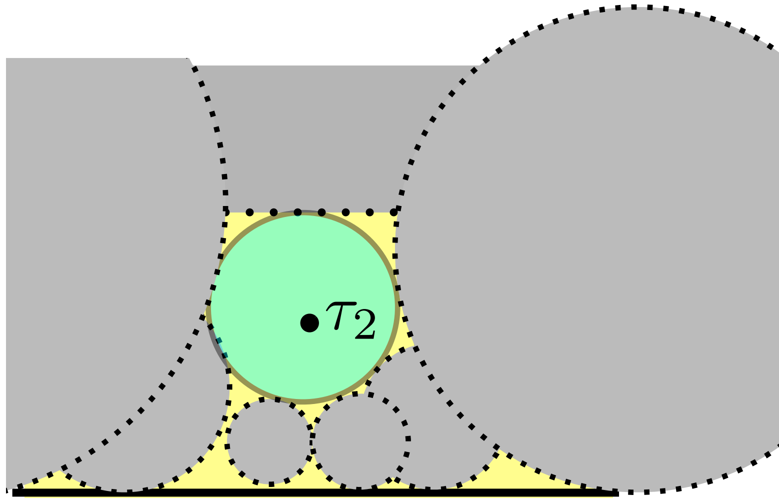
Hence

$$\operatorname{Im}\tau \le \operatorname{Im}\tau^* \frac{\iint_{E_\tau} \frac{|1+\mu|^2}{1-|\mu|^2} dxdy}{\operatorname{Im}\tau} \quad \left(\le \operatorname{Im}\tau^* \frac{\iint_{E_\tau} \frac{1+|\mu|}{1-|\mu|} dxdy}{\operatorname{Im}\tau} \le \operatorname{Im}\tau^* K(f)\right).$$

Lemma. Let $\tau_1, \tau_2 \in \mathbb{H}$. If there exists $L \geq 1$ such that

$$\text{for any } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}), \quad \operatorname{Im} A(\tau_1) \leq L \operatorname{Im} A(\tau_2),$$

then $\quad d_{\mathbb{H}}(\tau_1, \tau_2) \leq \log L.$



By $\quad \operatorname{Im} \tau \leq \operatorname{Im} \tau^* \dfrac{\iint_{E_\tau} \frac{|1+\mu|^2}{1-|\mu|^2} dxdy}{\operatorname{Im} \tau}$

and the change of basis of $\mathbb{Z} + \mathbb{Z}\tau$ by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$, which has an effect $\tau \mapsto \frac{a\tau+b}{c\tau+d}$ and $\mu \rightsquigarrow e^{i\theta}\mu$,

Torus version of Grötzsch-type ineq.

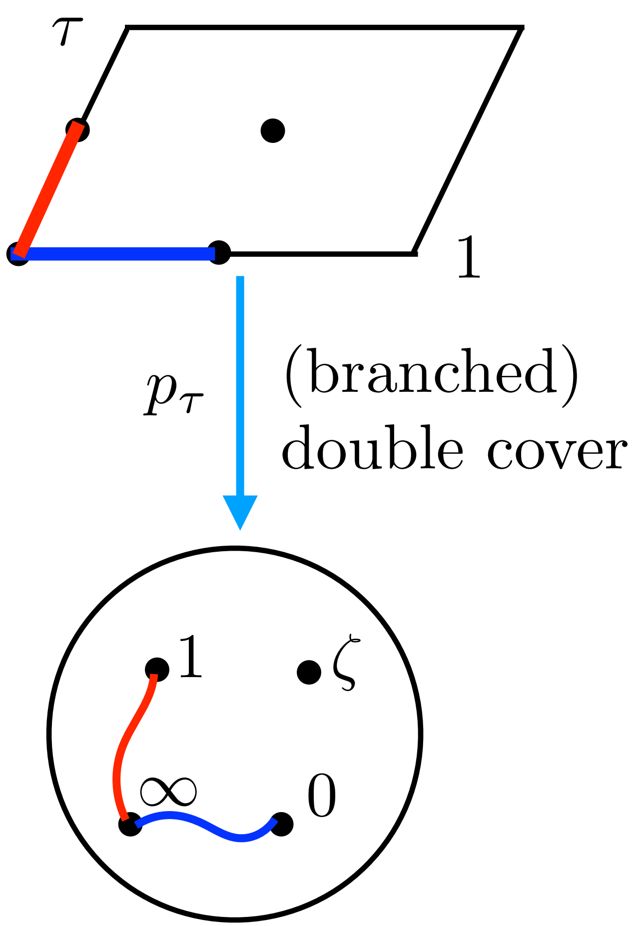
$$d_{\mathbb{H}}(\tau, \tau^*) \leq \log \overline{K}_f := \log \frac{\sup_{\theta \in \mathbb{R}} \iint_{E_\tau} \frac{|1+e^{i\theta}\mu_f(z)|^2}{1-|\mu_f(z)|^2} dx\, dy}{\operatorname{Im} \tau}$$

$$= \log\left(1 + 2|I_1| + 2I_2\right),$$

where $I_1 = \frac{1}{\operatorname{Im}\tau} \iint_{E_\tau} \frac{\mu(z) dxdy}{1-|\mu(z)|^2}$ and $I_2 = \frac{1}{\operatorname{Im}\tau} \iint_{E_\tau} \frac{|\mu(z)|^2 dxdy}{1-|\mu(z)|^2}$.

From torus to cross-ratio

$E_\tau = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ (via elliptic functions and elliptic modular function)



Torus E_τ $(\tau \in \mathbb{H})$
with generators $1, \tau$

Quadruple of points
with marking $\gamma_1,\ \gamma_2$

$\lambda(\tau) := \zeta = \frac{\zeta - 0}{1 - 0}\,\frac{1 - \infty}{0 - \infty}$
is the cross-ratio
of 4 branching pts

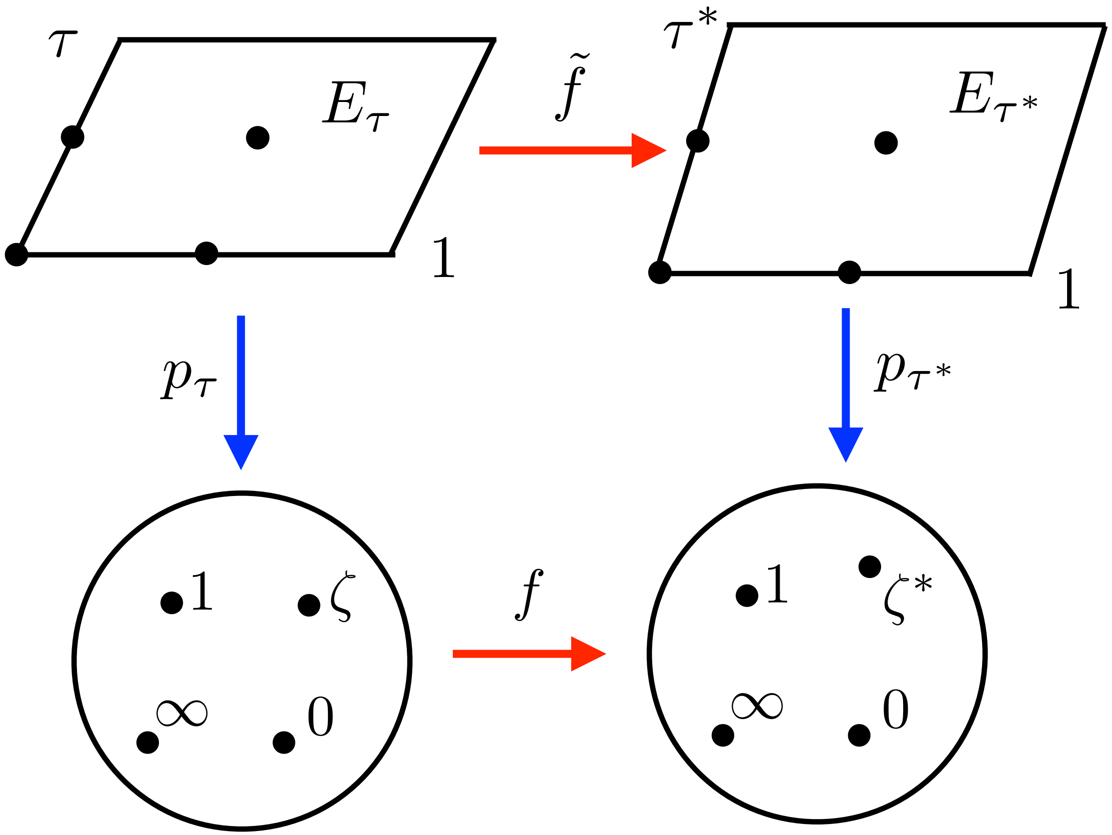
Differential equation $\quad \left(p'_\tau(w)\right)^2 = c_\tau\, p_\tau(w)(p_\tau(w) - 1)(p_\tau(w) - \zeta)$

c_τ is determined by $\frac{1}{2} = \int_\infty^0 \frac{dz}{\sqrt{c_\tau z(z-1)(z-\zeta)}}$

Fact. $\lambda : \mathbb{H} \to \mathbb{C} \setminus \{0, 1\},\ \tau \mapsto \zeta$ is a universal covering.

Elliptic modular function

Suppose $f : \mathbb{C} \to \mathbb{C}$ is K-qc and $(0, 1, \infty, \zeta) \mapsto (0, 1, \infty, \zeta^*)$ with marking.



$$du\,dv \qquad \tilde{\mu}(w)$$

$$z = p_\tau(w)$$

$$\left(\frac{dz}{dw}\right)^2 = c_\tau z(z-1)(z-\zeta)$$

$$\frac{dx\,dy}{|p_\tau'(w)|^2} \qquad \mu(z)\frac{\overline{p_\tau'(w)}}{p_\tau'(w)}$$

$$= \frac{dx\,dy}{|c_\tau z(z-1)(z-\zeta)|}$$

Grötzsch-type ineq for cross-ratio (Teichmüller)

$$d_{\mathbb{C}\setminus\{0,1\}}(\zeta,\zeta^*) = d_{\mathbb{H}}(\tau,\tau^*) \le \log(1+2|I_1|+2I_2) \le 2|I_1|+2I_2 = \frac{J(\mu,1,\zeta)}{J_*(1,\zeta)}$$

$\le \log K$ (classical)

where

$$J(\mu,z_1,z_2) = 2\left|\iint_{\mathbb{C}} \frac{\mu}{1-|\mu|^2}\varphi_{z_1,z_2}(z)dxdy\right| + 2\iint_{\mathbb{C}} \frac{|\mu|^2}{1-|\mu|^2}|\varphi_{z_1,z_2}(z)|dxdy,$$

$$J_*(z_1,z_2) = \iint_{\mathbb{C}} |\varphi_{z_1,z_2}(z)|dxdy \text{ with } \varphi_{z_1,z_2}(z) = \frac{z_1}{z(z-z_1)(z-z_2)}.$$

Proof of Key inequality

Lemma. For $L = \log K$, $0 < \exists\, \delta_1 < 1$, $\exists\, C_0 > 0$ such that if $d_{\mathbb{C}\setminus\{0,1\}}(\zeta_1, \zeta_2) \le L$ and $0 < |\zeta_1| < \delta_1$, then

$$|\log\zeta_1 - \log\zeta_2|_{\mathbb{C}/2\pi i\mathbb{Z}} \le C_0\, d_{\mathbb{C}\setminus\{0,1\}}(\zeta_1, \zeta_2)\log\frac{1}{|\zeta_1|}.$$

Idea of pf. $ds \sim \frac{|dz|}{|z|\log\frac{1}{|z|}}$ near 0, hence $\frac{1}{C_1}\log\frac{1}{|\zeta_1|} \le \log\frac{1}{|\zeta_2|} \le C_1\log\frac{1}{|\zeta_1|}$.

$|\log\zeta_1 - \log\zeta_2|_{\mathbb{C}/2\pi i\mathbb{Z}} \le \int_{\zeta_1}^{\zeta_2}\left|\frac{dz}{z}\right| \le C_0\int_{\zeta_1}^{\zeta_2} ds\,\log\frac{1}{|\zeta_1|} = C_0\, d_{\mathbb{C}\setminus\{0,1\}}(\zeta_1, \zeta_2)\log\frac{1}{|\zeta_1|}$.

Suppose $f : \mathbb{C} \to \mathbb{C}$ K-quasiconfromal , $f(0) = 0$ and $0 < |z_2| \le \delta_1|z_1|$.

Set $\zeta_1 = \frac{z_2}{z_1}$ and $\zeta_2 = \frac{f(z_2)}{f(z_1)}$. Then

$$\left|\log\frac{f(z_1)}{z_1} - \log\frac{f(z_2)}{z_2}\right|_{\mathbb{C}/2\pi i\mathbb{Z}} = |\log\zeta_1 - \log\zeta_2|_{\mathbb{C}/2\pi i\mathbb{Z}} \le C_0\, d_{\mathbb{C}\setminus\{0,1\}}(\zeta_1, \zeta_2)\log\frac{1}{|\zeta_1|}$$

$$\le C_0\frac{J(\mu, z_1, z_2)}{J_*(z_1, z_2)}\log\frac{1}{|\zeta_1|} \le C J(\mu, z_1, z_2).$$

Here $J_*(z_1, z_2) \ge C_2\log\frac{|z_1|}{|z_2|}$, since $\varphi_{z_1,z_2}(z) \sim -\frac{1}{z^2}$ for $|z_2| \ll |z| \ll |z_1|$.

$\left(J(\mu, z_1, z_2) = 2\left|\iint_{\mathbb{C}}\frac{\mu}{1-|\mu|^2}\varphi_{z_1,z_2}(z)dxdy\right| + 2\iint_{\mathbb{C}}\frac{|\mu|^2}{1-|\mu|^2}|\varphi_{z_1,z_2}(z)|dxdy,\ \ \varphi_{z_1,z_2}(z) = \frac{z_1}{z(z-z_1)(z-z_2)}.\right)$

Other applications of Key inequality

Theorem Let $f : \mathbb{C} \to \mathbb{C}$ be a quasiconformal mapping and suppose that for some $\beta > 0$,

$$I(r) = \iint_{\{z:|z|<r\}} \frac{|\mu(z)|}{1-|\mu(z)|^2} \frac{dx\,dy}{|z|^2} \text{ is finite and has order } O(r^\beta)\ (r \searrow 0).$$

Then for any $0 < \alpha < \frac{\beta}{2+\beta}$, f is $C^{1+\alpha}$-conformal at 0, i.e.

$$f(z) = f(0) + f'(0)z + \varepsilon_f(z) \quad \text{with } \varepsilon_f(z) = O(|z|^{1+\alpha}) \ \text{ as } z \to 0.$$

cf. Schatz, McMullen

Theorem (with D. Marti Pete) There exists a transcendental entire function of finite order with bounded singular values such that it has an oscillating wandering domain.

cf. Bishop's construction using qc-folding

Estimate on $\log \frac{\phi(z)}{z}$ for a qc ϕ whose μ_ϕ has support in disks B_m's such that radius$(B_m) \ll d(0, B_m)$ and B_m's are far aprt from each other.

Differentiability with respect to a parameter

Theorem. (Ahlfors-Bers) Suppose $\mu_t, \nu \in L^\infty(\mathbb{C})$ $(t \in [0, t_0])$, $\mu_0 \equiv 0$,

$$||\mu_t(z) - t\nu(z)||_\infty = o(t) \quad (t \to 0) \quad \text{(differentiable in } L^\infty \text{ sense)}.$$

Let $f_t : \mathbb{C} \to \mathbb{C}$ be the quasiconformal mapping satisfying $\mu_{f_t} = \mu_t(z)$ a.e. and $f_t(0) = 0$, $f_t(1) = 1$. Then for $\zeta \in \mathbb{C} \setminus \{0, 1\}$, $f_t(\zeta)$ is differentiable with respect to t at $t = 0$ and the derivative is given by

$$\left.\frac{\partial f_t(\zeta)}{\partial t}\right|_{t=0} = -\frac{\zeta(\zeta - 1)}{\pi} \iint_{\mathbb{C}} \frac{\nu(z)dxdy}{z(z-1)(z-\zeta)}.$$

The classical proof relies on heavy machinery of the measurable Riemann mapping theorem using singular integral operators and Calderón-Zygmund ineq.

We derive the parametric differentiability with a weaker assumption via the variation of cross-ratios.

Theorem. Suppose $||\mu_t||_\infty = o(\sqrt{t})$ and $\iint_{\mathbb{C}} \frac{\mu_t(z)dxdy}{z(z-1)(z-\zeta)} = t\tilde{J}_0 + o(t)$ $(t \to 0)$. Then $f_t(\zeta)$ is differentiable with respect to t at $t = 0$ and the derivative is given by $\left.\frac{\partial f_t(\zeta)}{\partial t}\right|_{t=0} = -\frac{\zeta(\zeta-1)}{\pi}\tilde{J}_0$.

back to torus

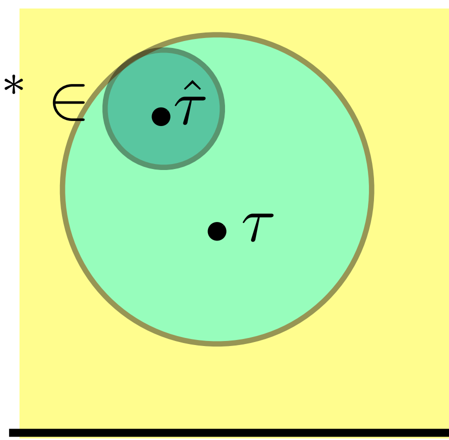
Theorem. Suppose $f : E_\tau \to E_{\tau^*}$ quasiconformal, $\mu = \mu_f$. Define

$$I_1 = \frac{1}{\operatorname{Im}\tau} \iint_{E_\tau} \frac{\mu(z)}{1-|\mu(z)|^2} dxdy, \quad I_2 = \frac{1}{\operatorname{Im}\tau} \iint_{E_\tau} \frac{|\mu(z)|^2}{1-|\mu(z)|^2} dxdy,$$

$$\hat\tau = \operatorname{Re}\tau + \operatorname{Im}\tau \frac{2\operatorname{Im} I_1 + i\sqrt{(1+2I_2)^2 - 4|I_1|^2}}{1 + 2\operatorname{Re} I_1 + 2I_2}, \quad \hat K = \sqrt{(1+2I_2)^2 - 4|I_1|^2},$$

$$\check\tau = \tau + 2i\operatorname{Im}\tau \frac{-I_1 + I_2^2 - |I_1|^2}{1 + 2\operatorname{Re} I_1 + 2I_2}, \quad \check R = 2\operatorname{Im}\tau \frac{I_2 + I_2^2 - |I_1|^2}{1 + 2\operatorname{Re} I_1 + 2I_2}.$$

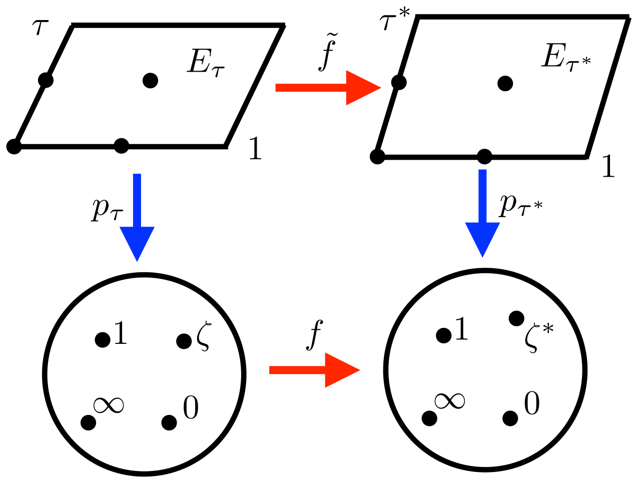
Then $\quad d_{\mathbb{H}}(\tau^*, \hat\tau) \le \log \hat K \quad$ and $\quad |\tau^* - \check\tau| \le \check R.$

$\tau^* \in$ 

Corollary. Suppose $f_s : E_\tau \to E_{\tau^*(s)}$ is a family of qc-mappings ($s \in [0, s_0]$). Let $I_1(s)$, $I_2(s)$ be corresponding integrals of $\mu_s = \mu_{f_s}$.
If $I_1(s) = s\tilde I_1 + o(s)$, $I_2(s) = o(s)$ $(s \to 0)$, then

$$\left.\frac{d\tau^*(s)}{ds}\right|_{s=0} = -2i\tilde I_1 \operatorname{Im}\tau.$$

variation of cross-ratio



Suppose $f_s : \mathbb{C} \to \mathbb{C}$ is a family of qc-mappings, with $f_s(0) = 0$, $f_s(1) = 1$, $\mu_s = \mu_{f_s}$ $(s \in [0, s_0])$.

Let $\tilde{f}_s : E_\tau \to E_{\tau^*(s)}$ be a lift, $\tilde{\mu}_s = \mu_{\tilde{f}_s}$.

Fix $\zeta \in \mathbb{C} \setminus \{0, 1\}$. Then $f_s(\zeta) = \lambda(\tau^*(s))$.

Define $J_1(s) = \iint_{\mathbb{C}} \frac{\mu_s(z)}{1-|\mu_s(z)|^2} \frac{dxdy}{c_\tau z(z-1)(z-\zeta)}$,

$$J_2(s) = \iint_{\mathbb{C}} \frac{|\mu_s(z)|^2}{1-|\mu_s(z)|^2} \frac{dxdy}{|c_\tau z(z-1)(z-\zeta)|},$$

$$J_* = \iint_{\mathbb{C}} \frac{dxdy}{|c_\tau z(z-1)(z-\zeta)|} = \frac{\operatorname{Im}\tau}{2}.$$

Then $I_1(s) = J_1(s)/J_*$, $I_2(s) = J_2(s)/J_*$.

By Corollary, if $J_1(s) = s\tilde{J}_1 + o(s)$, $J_2(s) = o(s)$ $(s \to 0)$,

$$\left.\frac{\partial f_s(\zeta)}{\partial s}\right|_{s=0} = \lambda'(\tau)\frac{d\tau^*}{ds}(0) = \lambda'(\tau)(-2i\tilde{J}_1\frac{2}{\operatorname{Im}\tau})\operatorname{Im}\tau = -4i\lambda'(\tau)\tilde{J}_1.$$

This is the case if $||\mu_s||_\infty = o(\sqrt{s})$ and $\iint \frac{\mu_s(z)dxdy}{z(z-1)(z-\zeta)} = s\tilde{J}_0 + o(s)$,

therefore $\left.\frac{\partial f_s(\zeta)}{\partial s}\right|_{s=0} = -4i\frac{\lambda'(\tau)}{c_\tau}\tilde{J}_0.$

In order to determine the coefficient, we only need to consider one example. Let $f_s(z) = z + sh(z)$, where $h(z)$ is smooth and supported on a small disk around ζ and $h(\zeta) = 1$. Then $f_s(\zeta) = \zeta + s$ and $\mu_s(z) = s\, h_{\overline{z}}(z) + O(s^2)$, hence by the previous formula and Pompeiu formula,

$$1 = \frac{\partial f_s(\zeta)}{\partial s} = -4i\frac{\lambda'(\tau)}{c_\tau}\iint_{\mathbb{C}} \frac{h_{\overline{z}}(z)dxdy}{z(z-1)(z-\zeta)} = 4i\frac{\lambda'(\tau)}{c_\tau}\frac{\pi}{\zeta(\zeta-1)}.$$

Hence we obtain the formula $\left.\frac{\partial f_t(\zeta)}{\partial t}\right|_{t=0} = -\frac{\zeta(\zeta-1)}{\pi}\tilde{J}_0.$

As a byproduct, we obtain:

Integral-differential equation for elliptic modular function $\lambda(\tau)$

$$\lambda'(\tau) = \frac{1}{\pi i}\lambda(\tau)(\lambda(\tau)-1)\left(\int_\infty^0 \frac{dz}{\sqrt{z(z-1)(z-\lambda(\tau))}}\right)^2.$$ reference??

Agard's formula for the Poincaré metric $\rho(z)|dz|$ of $\mathbb{C}\setminus\{0,1\}$:

$$\frac{1}{\rho(\zeta)} = \frac{|\zeta(\zeta-1)|}{2\pi}\iint_{\mathbb{C}} \frac{dxdy}{|z(z-1)(z-\zeta)|}.$$

Outline of Proof of Main Estimate

Lemma. Let $\tau_1, \tau_2 \in \mathbb{H}$. If there exists $L \geq 1$ such that

$$\text{for any } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}), \text{ or } \Gamma(2) \quad \operatorname{Im} A(\tau_1) \leq L \operatorname{Im} A(\tau_2),$$

then $\quad d_{\mathbb{H}}(\tau_1, \tau_2) \leq \log L.$

Look for $\hat{\tau}$ and L such that for all c, d

$$\operatorname{Im} A(\hat{\tau}) \leq L \operatorname{Im} A(\tau^*) \iff \frac{|c\tau^* + d|^2}{\operatorname{Im} \tau^*} \leq L \frac{|c\hat{\tau} + d|^2}{\operatorname{Im} \hat{\tau}}.$$

On the other hand, Grötzsch inequality after change of basis gives:

$$\frac{|c\tau^* + d|^2}{\operatorname{Im} \tau^*} \leq \left(\frac{|c\tau + d|^2}{\operatorname{Im} \tau} \right)^2 \iint_{\mathbb{E}_\tau} \frac{|1 + \mu(z) \frac{c\overline{\tau}+d}{c\tau+d}|^2}{1 - |\mu(z)|^2} \frac{dx\,dy}{|c\tau + d|^2}$$

$$= \frac{1}{(\operatorname{Im} \tau)^2} \iint_{\mathbb{E}_\tau} \frac{|(c\tau + d) + (c\overline{\tau} + d)\mu(z)|^2}{1 - |\mu(z)|^2} dx\,dy.$$

We only need to find $\hat{\tau}$ and L so that the right hand side of two inequalities coincide, i.e. just match coefficients of c^2, cd, d^2.

Such $\hat{\tau}$ and L can be expressed in terms of

$I_1 = \frac{1}{\operatorname{Im} \tau} \iint_{E_\tau} \frac{\mu(z)}{1 - |\mu(z)|^2} dxdy$ and $I_2 = \frac{1}{\operatorname{Im} \tau} \iint_{E_\tau} \frac{|\mu(z)|^2}{1 - |\mu(z)|^2} dxdy.$

Example

$\mu_s(z) = s^{1/2}\lambda_s(z) + s\nu(z)$
$\lambda_s(z)$ fast oscillating, e.g. $\lambda_s(z) = e^{ip(s)x}$ with $p(s) \to \infty$ fast