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The polarization with respect to hypersphere and its applications.

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Introduction

The method of symmetrization has a lot of applications in the geometric theory of functions. The first geometric transformation bearing the name symmetrization was introduced by Steiner in 1836. It is now well known that many symmetrizations can be obtained as a limit of the polarizations with respect to a hyperplane [1], [2]. This transformation was introduced for sets by Wolontis in 1952, the term polarization was suggested by Dubinin in 1985.

1.V.N. Dubinin. Condenser capacities and symmetrization in geometric function theory. Birkhauser Basel, 2014.

2. A. Yu. Solynin, Continuous symmetrization via polarization, Algebra i Analiz, **24**:1 (2012), 157–222; St. Petersburg Math. J., **24**:1 (2013), 117–166.

However polarization with respect to hypersphere has not receive much attention . We will discuss the basic properties of the polarization with respect to hypersphere. Applications include some isoperimetric inequalities for conformal capacity of condensers, estimates of conformal metric and distortion theorems for quasiregular mappings in a annulus [3], inequalities for n-harmonic Levitskii radius [4], [5].

3. M. Vuorinen, Conformal geometry and quasiregular mappings, Lecture Notes in Mathematics, 1988, Springer-Verlag.

4.B. Levitskiĭ, 3. Reduced p-modulus and the interior p-harmonic radius, Dokl. Akad. Nauk SSSR **316 (4)** (1991) 812–815 (in Russian); translation in: Soviet Math. Dokl. **43(1)** (1991) 189–192.

5. W. Wang N-Capacity, N-harmonic radius and N-harmonic transplantation, J. Math. Anal. Appl. **327**:1 (2007), 155–174.

Content

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- N – harmonic Levitskii radius
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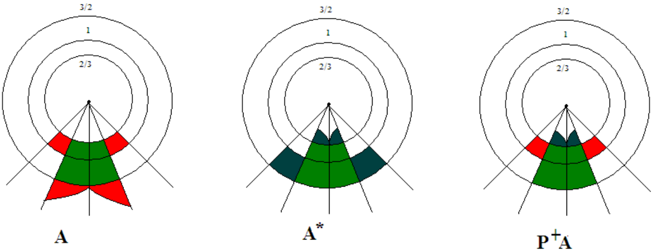
- Isoperimetric inequalities for conformal capacity of condensers
- Estimates of conformal metric in the annulus
- Distortion theorem for quasiregular mappings in the annulus

For $A \subset \overline{\mathbb{R}}^n$ denote by $A^+ = A^+(\rho) = \{x \in A : \mid x - a \mid \leq \rho\}$, $A^- = A^-(\rho) = \{x \in A : \mid x - a \mid \geq \rho\}$, $A^* = \{x \in \overline{\mathbb{R}}^n : x^* \in A\}$, where x^*, x are symmetric with respect to hypersphere $S(a, \rho) = \{x \in \overline{\mathbb{R}}^n : |x| = \rho\}$, $x^* = \rho^2 x / |x|^2$.

$$P^+A = (A \cap A^*)^- \cup (A \cup A^*)^+, \;\; P^-A = (A \cap A^*)^+ \cup (A \cup A^*)^-.$$

Polarization with respect to hypersphere $S(a, \rho)$ is the transformation $A \to P^+A$ (or $A \to P^-A$).



The polarization with respect to hypersphere

Assume that $E \subset \overline{\mathbb{R}}^n$ is closed set, D is open set, $E \subset D$, Γ is family of curves joining E with ∂D in D.

Then we can define conformal $n - capacity$ of condenser $C = (E, D)$ as $n - modulus$ of Γ

$$cap_n\, C = M_n(\Gamma) = \inf \int_{\mathbb{R}^n} \rho^n dx,$$

where the inf is taken over all Borel functions $\rho : \mathbb{R}^n \to [0, \infty]$, such that the inequality $\int_\gamma \rho ds \geq 1$ holds for every curve $\gamma \in \Gamma$.

Polarization of condenser C with respect to hypersphere $S(a, \rho)$ is the transformation $C = (E, D) \to PC = (P^+E, P^+D)$.

- (Principle of polarization)

$$cap_n\, C \geq cap_n\, PC$$

The conformal radius (in the complex plane) seems to appear for the first time in the proof of the Riemann mapping theorem. The conformal radius has many applications in the geometric function theory. A whole section of the famous collection Problems and Theorems in Analysis by Pólya and Szegő is devoted to its properties and actual computation. A generalization of planar conformal radius to higher dimensional domains $D \subset \overline{\mathbb{R}}^n$ is known as n-harmonic radius introduced by Levitskii. Its definition deals with n-harmonic Green function. Another way to define *n-harmonic radius* is using the following identity

$$\log R_n(a, D) = \lambda_n M_n(t, a, D)^{\frac{1}{1-n}} + \log(t) + o(1), \quad t \to 0.$$

where $M_n(t, a, D)$ is the n-modulus of the curve family $\Gamma(t, a, D)$, which consists of all curves joining the hypersphere $S(a, t)$ and ∂D in D, $\lambda_n = (n\omega_n)^{\frac{1}{p-1}}$, ω_n is the volume of the ball $B(0, 1)$.

The point of greatest conformal radius is called conformal center of D. For the unit disk U we have

$$r(U,z) = 1 - |z|^2.$$

Hence $z = 0$ is the conformal center. Now we consider a domain which is star-shaped with respect to origin. For example,

$$D = U(0.5;1),\ \ r(U,0.5) = 1,\ \ r(U,0) = 0.75$$

Hence $z = 0$ is not conformal center.

Applying the polarization with respect to hypersphere, we obtain the following result [6].

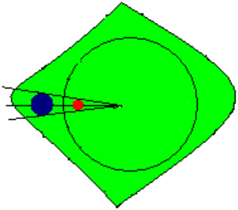
6. E. G. Prilepkina, On the n-harmonic radius of domains in the n-dimensional Euclidean space, Far Eastern Math. J., **17**:2 (2017), 246–256 (in russian).

Theorem

Let $D \subset \overline{\mathbb{R}}^n$ be a star-shaped domain with respect to origin, $a,\ b\ \in D,\ \ 0 < |a| < |b|, a/|a| = b/|b|$. *Then*

$$R_n(D, b)/|b| \leq R_n(D, a)/|a|.$$

Or the function $f(t) = R_n(D, ta)/t$ *decreases on* $(0, \infty)$.



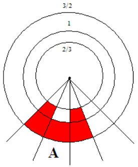
Let A be closed subset $\overline{\mathbb{R}}^n \setminus \{0\}$, $v \in S(0,1)$, $K(v)$ is a ray $K(v) = \{tv : t \geq 0\}$, $I(v,A) = K(v) \cap A$.

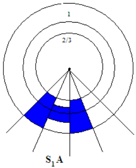
Let $\mu_v = \exp(0.5 \int_{I(v,A)} \frac{dt}{t})$ be the log-measure $K(v) \cap A$. We define $K(v,A) = \emptyset$, if $K(v) \cap A = \emptyset$,

$$K(v,A) = K(v) \cap \{x \in \overline{\mathbb{R}}^n : \rho\mu_v^{-1} \leq | x | \leq \rho\mu_v\}, \text{ if } K(v) \cap A \neq \emptyset.$$

Symmetrization of A with respect to hypersphere $S(0,\rho)$ is the transformation $A \to S_\rho A$,

$$S_\rho A = \cup_{v \in S(0,1)} K(v,A).$$





The symmetrization with respect to hypersphere

Let A be an open subset $\overline{\mathbb{R}}^n \setminus \{0\}$. Then

$K(v, A) = \emptyset$, if $K(v) \cap A = \emptyset$,

$K(v, A) = K(v) \cap \{x \in \overline{\mathbb{R}}^n \colon \rho\mu_v^{-1} <\mid x \mid< \rho\mu_v\}$, if $K(v) \cap A \neq \emptyset$;

$S_\rho A = \cup_{v \in S(0,1)} K(v, A)$.

Symmetrization of condenser $C = (E, D)$ with respect to hypersphere $S(0, \rho)$ is the transformation $C \to S_\rho C = (S_\rho E, S_\rho D)$.

Theorem (Dubinin-P.)

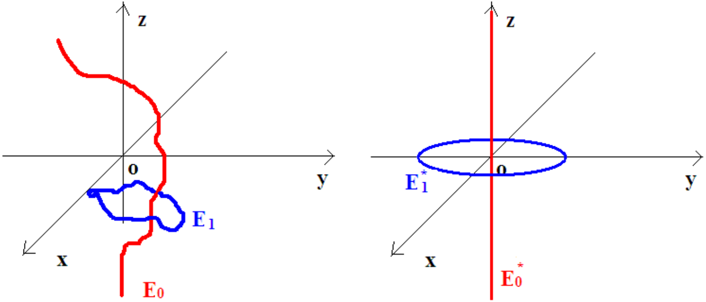
If $0 \in \overline{\mathbb{R}}^n \setminus D$, $\infty \in \overline{\mathbb{R}}^n \setminus D$,

$$cap_n\, C \geq cap_n\, S_\rho C$$

Open problem (Aseev): Let $C=(E_0,\overline{\mathbb{R}}^3\setminus E_1)$ be a condenser in $\overline{\mathbb{R}}^3$ with plates E_0 and E_1, where E_1 is a closed curves, E_0 is another curve passing through "the interiory"of E_1. Show that

$$cap_n\, C \geq cap_n\, C^*,$$

where $C^*=(E_0^*,\overline{\mathbb{R}}^3\setminus E_1^*)$, E_0^* is the z-axis, E_1^* is a circle in xOy - plane with center at the origin.

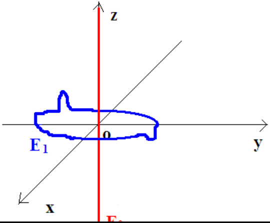


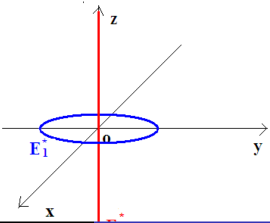
Theorem

Particular case of Aseev problem: *Let* $C = (E_0, \overline{\mathbb{R}}^3 \setminus E_1)$ *be a condenser in* $\overline{\mathbb{R}}^3$ *with plates* E_0 *and* E_1*, where* E_1 *is a closed curves,* E_0 *is another curve passing through "the interiory"of* E_1*, but* E_0 *is the z-axis. Then*

$$cap_n\, C \geq cap_n\, C^*,$$

where $C^* = (E_0^*, \overline{\mathbb{R}}^3 \setminus E_1^*)$*,* E_0^* *is the z-axis,* E_1^* *is a circle in xOy - plane with center at the origin.*





Isoperimetric inequalities for conformal capacity of condensers

Let $K_r = \{x \in \overline{\mathbb{R}}^n : r < |x| < 1/r\}$ be annulus,

$E(s)$ be spherical cap on the hypersphere $S(0,1)$ centered at the point $(1,0,...,0)$ and $m_{n-1}(E(s)) = s$,

$C(r,s) = (E(s), K_r)$,

$\lambda(r,s) = cap_n\, C(r,s)$.

Theorem

Assume that $D \subset K_r$, $(n-1)$-measure of projection E on the sphere $S(0,1)$ is not less than s. Then for condenser $C = (E,D)$ we have

$$cap_n C \geq \lambda(r,s).$$

Equality is achieved for $C(r,s)$.

Isoperimetric inequalities for conformal capacity of condensers

Let $E(a,r)=\{x\in\overline{\mathbb{R}}^n:|x|\leq r\}$ be a ball,

$C(\rho)=(K(e_n),\overline{\mathbb{R}}^n\setminus(E(-e_1,\rho)\cup E(e_1,\rho)))$, $e_1=(1,0,...,0)$, $e_n=(0,0,...,1)$, $0<\rho<1$.

Theorem

If $(E(-e_1,\rho)\cup E(e_1,\rho))\subset\overline{\mathbb{R}}^n\setminus D$ and E is a curve joining 0 and ∞, then for $C=(E,D)$ we have inequality

$$cap_n C\geq cap_n C(\rho).$$

For a proper subdoman $G \subset \overline{\mathbb{R}}^n$ and for all $x, y \in G$ we define conformal metric $\mu_G(x, y)$,

$$\mu_G(x, y) = \inf_{C_{xy}} M_n(\triangle(C_{xy}, \partial G; G)),$$

where infimum is taken over all continua C_{xy} such that $C_{xy} = \gamma[0, 1]$ and γ is a curve with $\gamma(0) = x$ and $\gamma(1) = y$. Here $\triangle(E, F, G)$ is family of all closed non-constant curves joining E and F in G, $M_n(\triangle(E, F, G))$ is its conformal module.

Property 1. It follows from definition that in the case $D \subset G$ we have

$$\mu_G(x, y) \leq \mu_D(x, y)$$

for all $x, y \in D$.

Property 2. If $f : G \to \overline{\mathbb{R}}^n$ is a non-constant quasiregular mapping, then

$$\mu_{fG}(f(\mathbf{x}),\ f(y)) \leq K_I(f)\mu_G(x,\ y); \qquad x,\ y \in G.$$

If f is quasiconformal mappings, then

$$\mu_{fG}(f(x),\ f(y)) \geq \mu_G(x, \mathbf{y})/K_0(f)$$

For unit ball $B^n = B(0,1) = \{x \in \overline{\mathbb{R}}^n : |x| < 1\}$, we have

$$\mu_{B^n}(x,y) = \gamma_n(1/th(0.5\rho(x,y)),$$

where $\gamma_n(s)$ is capacity of Grotzsh ring, $\rho(x,y)$ is hyperbolic metric. If we have

$$\mu_G(x,y) = M_n(\triangle(\gamma_{xy}, \partial G; G)),$$

we'll call γ_{xy} geodesic. We know the geodesic in the ball B^n because we can apply a conformal mapping and spherical symmetrization.

But what about the annulus?

By using the polarization with respect to hypersphere for annulus $K_r = \{r < |x| < 1/r\}$ and $x, y \in K_r$, $|x| = |y| = 1$, we proved that

$$\mu_{K_r}(x, y) = M_n(\triangle(\lambda_{xy}, \partial K_r; K_r)),$$

where λ_{xy} joins x with y and it is the smallest arc of the unit circle in the 2-dimensional plane Oxy.

For fixed α, $0 \leq \alpha \leq \pi$ we define l_α as λ_{xy}, where $x = (\cos\alpha/2, \sin\alpha/2, 0, ..., 0)$, $y = (\cos\alpha/2, -\sin\alpha/2, 0, ..., 0)$.

For fixed r, $0 < r < 1$, we define $u_{n,r}(\alpha)$, $0 \leq \alpha \leq \pi$, $v_{n,r}(s)$, $r < s \leq 1$, as modulus of curves family:

$$u_{n,r}(\alpha) = M_n(\Delta(l_\alpha, S(0,r); K(r,1)));$$

$$v_{n,r}(s) = M_n(\Delta([se_1, e_1], S(0,r); K(r,1))).$$

Theorem

Assume that $x \in K_r$, $y \in K_r$, $|x| \leq |y|$, α_{xy} is angle between x, y. Then

$$\mu_{K_r}(x,y) \geq 2u_{n,r}(\alpha_{xy}); \tag{1}$$

$$\mu_{K_r}(x,y) \geq 2v_{n,r}(\sqrt{|x|/|y|}). \tag{2}$$

The equality into (1) is attend when $|x| = |y| = 1$. And in the case of symmetric points x, y with respect to $S(0,1)$ we have equality into (2).

The distortion theorem for quasiregular mappings in the annulus

Theorem

Assume that $x \in S(0,1)$, $y \in S(0,1)$, $f(x)$ is K-quasiregular mapping of K_r, $f(K_r) \subset K_{r_1}$. Then

$$u_{n,r_1}(\alpha_{f(x)f(y)}) \leq K u_{n,r}(\alpha_{xy}),$$

where α_{xy} is the angle between x, y and $\alpha_{f(x)f(y)}$ is the angle between $f(x)$, $f(y)$.

It is possible to compute $u_{n,r}(x,y)$ for complex plane.

Thank you for attention!