Gleason Parts for Uniform Algebras without Analytic Discs

D. Papathanasiou joint work with A. Izzo
University of Mons

July, 2018

X = compact Hausdorff space

$C(X) = \{f : X \to \mathbb{C}, \text{continuous}\}$ endowed with $\|f\|_\infty = \sup\{|f(x)| : x \in X\}$

A **uniform algebra** A on X is a uniformly closed subalgebra of $C(X)$ that contains the constants and separates the points on X.

The **maximal ideal space** of A is

$$\mathfrak{M}_A = \{\phi : A \to \mathbb{C}, \phi \neq 0, \text{multiplicative, linear}\}$$

topologized with the relative weak* topology.

$\mathfrak{M}_A$= compact subset of $S(0,1) \subset A^*$

$X \hookrightarrow \mathfrak{M}_A$, by $x \mapsto \phi_x, \phi_x(f) = f(x)$.

$A \cong \hat{A} \subset C(\mathfrak{M}_A)$, by $f \mapsto \hat{f}, \hat{f}(\phi) = \phi(f)$.

Notice, for $x \in X, \hat{f}(\phi_x) = \phi_x(f) = f(x)$,

so we may think $X \subset \mathfrak{M}_A$, $A = \hat{A}$ and the functions of A extend from X to $\mathfrak{M}_A$.

X = compact Hausdorff space
$C(X) = \{f : X \to \mathbb{C}, \text{continuous}\}$ endowed with $\|f\|_\infty = \sup\{|f(x)| : x \in X\}$
A **uniform algebra** A on X is a uniformly closed subalgebra of $C(X)$ that contains the constants and separates the points on X.
The **maximal ideal space** of A is

$$\mathfrak{M}_A = \{\phi : A \to \mathbb{C}, \phi \neq 0, \text{multiplicative, linear}\}$$

topologized with the relative weak* topology.
$\mathfrak{M}_A$= compact subset of $S(0,1) \subset A^*$
$X \hookrightarrow \mathfrak{M}_A$, by $x \mapsto \phi_x, \phi_x(f) = f(x)$.
$A \cong \hat{A} \subset C(\mathfrak{M}_A)$, by $f \mapsto \hat{f}, \hat{f}(\phi) = \phi(f)$.
Notice, for $x \in X, \hat{f}(\phi_x) = \phi_x(f) = f(x)$,
so we may think $X \subset \mathfrak{M}_A$, $A = \hat{A}$ and the functions of A extend from X to $\mathfrak{M}_A$.

X = compact Hausdorff space
$C(X) = \{f : X \to \mathbb{C}, \text{continuous}\}$ endowed with $\|f\|_\infty = \sup\{|f(x)| : x \in X\}$
A **uniform algebra** A on X is a uniformly closed subalgebra of $C(X)$ that contains the constants and separates the points on X.
The **maximal ideal space** of A is

$$\mathfrak{M}_A = \{\phi : A \to \mathbb{C}, \phi \neq 0, \text{multiplicative, linear}\}$$

topologized with the relative weak* topology.
$\mathfrak{M}_A$= compact subset of $S(0,1) \subset A^*$
$X \hookrightarrow \mathfrak{M}_A$, by $x \mapsto \phi_x, \phi_x(f) = f(x)$.
$A \cong \hat{A} \subset C(\mathfrak{M}_A)$, by $f \mapsto \hat{f}, \hat{f}(\phi) = \phi(f)$.
Notice, for $x \in X, \hat{f}(\phi_x) = \phi_x(f) = f(x)$,
so we may think $X \subset \mathfrak{M}_A$, $A = \hat{A}$ and the functions of A extend from X to $\mathfrak{M}_A$.

X = compact Hausdorff space
$C(X) = \{f : X \to \mathbb{C}, \text{continuous}\}$ endowed with $\|f\|_\infty = \sup\{|f(x)| : x \in X\}$
A **uniform algebra** A on X is a uniformly closed subalgebra of $C(X)$ that contains the constants and separates the points on X.
The **maximal ideal space** of A is

$$\mathfrak{M}_A = \{\phi : A \to \mathbb{C}, \phi \neq 0, \text{multiplicative, linear}\}$$

topologized with the relative weak* topology.
$\mathfrak{M}_A$= compact subset of $S(0,1) \subset A^*$
$X \hookrightarrow \mathfrak{M}_A$, by $x \mapsto \phi_x, \phi_x(f) = f(x)$.
$A \cong \hat{A} \subset C(\mathfrak{M}_A)$, by $f \mapsto \hat{f}, \hat{f}(\phi) = \phi(f)$.
Notice, for $x \in X, \hat{f}(\phi_x) = \phi_x(f) = f(x)$,
so we may think $X \subset \mathfrak{M}_A$, $A = \hat{A}$ and the functions of A extend from X to $\mathfrak{M}_A$.

X = compact Hausdorff space
$C(X) = \{f : X \to \mathbb{C}, \text{continuous}\}$ endowed with $\|f\|_\infty = \sup\{|f(x)| : x \in X\}$
A **uniform algebra** A on X is a uniformly closed subalgebra of $C(X)$ that contains the constants and separates the points on X.
The **maximal ideal space** of A is

$$\mathfrak{M}_A = \{\phi : A \to \mathbb{C}, \phi \neq 0, \text{multiplicative, linear}\}$$

topologized with the relative weak* topology.
$\mathfrak{M}_A$= compact subset of $S(0,1) \subset A^*$
$X \hookrightarrow \mathfrak{M}_A$, by $x \mapsto \phi_x, \phi_x(f) = f(x)$.
$A \cong \hat{A} \subset C(\mathfrak{M}_A)$, by $f \mapsto \hat{f}, \hat{f}(\phi) = \phi(f)$.
Notice, for $x \in X, \hat{f}(\phi_x) = \phi_x(f) = f(x)$,
so we may think $X \subset \mathfrak{M}_A$, $A = \hat{A}$ and the functions of A extend from X to $\mathfrak{M}_A$.

X = compact Hausdorff space
$C(X) = \{f : X \to \mathbb{C}, \text{continuous}\}$ endowed with $\|f\|_\infty = \sup\{|f(x)| : x \in X\}$
A **uniform algebra** A on X is a uniformly closed subalgebra of $C(X)$ that contains the constants and separates the points on X.
The **maximal ideal space** of A is

$$\mathfrak{M}_A = \{\phi : A \to \mathbb{C}, \phi \neq 0, \text{multiplicative, linear}\}$$

topologized with the relative weak* topology.
$\mathfrak{M}_A$= compact subset of $S(0,1) \subset A^*$
$X \hookrightarrow \mathfrak{M}_A$, by $x \mapsto \phi_x, \phi_x(f) = f(x)$.
$A \cong \hat{A} \subset C(\mathfrak{M}_A)$, by $f \mapsto \hat{f}, \hat{f}(\phi) = \phi(f)$.
Notice, for $x \in X, \hat{f}(\phi_x) = \phi_x(f) = f(x)$,
so we may think $X \subset \mathfrak{M}_A$, $A = \hat{A}$ and the functions of A extend from X to $\mathfrak{M}_A$.

$X =$ compact Hausdorff space
$C(X) = \{f : X \to \mathbb{C}, \text{continuous}\}$ endowed with $\|f\|_\infty = \sup\{|f(x)| : x \in X\}$
A **uniform algebra** A on X is a uniformly closed subalgebra of $C(X)$ that contains the constants and separates the points on X.
The **maximal ideal space** of A is

$$\mathfrak{M}_A = \{\phi : A \to \mathbb{C}, \phi \neq 0, \text{multiplicative, linear}\}$$

topologized with the relative weak* topology.
$\mathfrak{M}_A=$ compact subset of $S(0,1) \subset A^*$
$X \hookrightarrow \mathfrak{M}_A$, by $x \mapsto \phi_x, \phi_x(f) = f(x)$.
$A \cong \hat{A} \subset C(\mathfrak{M}_A)$, by $f \mapsto \hat{f}, \hat{f}(\phi) = \phi(f)$.
Notice, for $x \in X, \hat{f}(\phi_x) = \phi_x(f) = f(x)$,
so we may think $X \subset \mathfrak{M}_A$, $A = \hat{A}$ and the functions of A extend from X to $\mathfrak{M}_A$.

X = compact Hausdorff space
$C(X) = \{f : X \to \mathbb{C}, \text{continuous}\}$ endowed with $\|f\|_\infty = \sup\{|f(x)| : x \in X\}$
A **uniform algebra** A on X is a uniformly closed subalgebra of $C(X)$ that contains the constants and separates the points on X.
The **maximal ideal space** of A is

$$\mathfrak{M}_A = \{\phi : A \to \mathbb{C}, \phi \neq 0, \text{multiplicative, linear}\}$$

topologized with the relative weak* topology.
$\mathfrak{M}_A$= compact subset of $S(0,1) \subset A^*$
$X \hookrightarrow \mathfrak{M}_A$, by $x \mapsto \phi_x, \phi_x(f) = f(x)$.
$A \cong \hat{A} \subset C(\mathfrak{M}_A)$, by $f \mapsto \hat{f}, \hat{f}(\phi) = \phi(f)$.
Notice, for $x \in X, \hat{f}(\phi_x) = \phi_x(f) = f(x)$,
so we may think $X \subset \mathfrak{M}_A$, $A = \hat{A}$ and the functions of A extend from X to $\mathfrak{M}_A$.

X = compact Hausdorff space
$C(X) = \{f : X \to \mathbb{C}, \text{continuous}\}$ endowed with $\|f\|_\infty = \sup\{|f(x)| : x \in X\}$
A **uniform algebra** A on X is a uniformly closed subalgebra of $C(X)$ that contains the constants and separates the points on X.
The **maximal ideal space** of A is

$$\mathfrak{M}_A = \{\phi : A \to \mathbb{C}, \phi \neq 0, \text{multiplicative, linear}\}$$

topologized with the relative weak* topology.
$\mathfrak{M}_A$= compact subset of $S(0,1) \subset A^*$
$X \hookrightarrow \mathfrak{M}_A$, by $x \mapsto \phi_x, \phi_x(f) = f(x)$.
$A \cong \hat{A} \subset C(\mathfrak{M}_A)$, by $f \mapsto \hat{f}, \hat{f}(\phi) = \phi(f)$.
Notice, for $x \in X, \hat{f}(\phi_x) = \phi_x(f) = f(x)$,
so we may think $X \subset \mathfrak{M}_A$, $A = \hat{A}$ and the functions of A extend from X to $\mathfrak{M}_A$.

Example

For X a compact, T_2 space, $C(X)$ is a uniform algebra with $\mathfrak{M}_{C(X)} = X$.

The disc algebra

If $D = \{z \in \mathbb{C} : |z| < 1\}$ then

$$A(D) = \{f \in C(\bar{D}) : f \in H(D)\}$$

is a uniform algebra with $\mathfrak{M}_{A(D)} = \bar{D}$.

Example

For X a compact, T_2 space, $C(X)$ is a uniform algebra with $\mathfrak{M}_{C(X)} = X$.

The disc algebra

If $D = \{z \in \mathbb{C} : |z| < 1\}$ then

$$A(D) = \{f \in C(\bar{D}) : f \in H(D)\}$$

is a uniform algebra with $\mathfrak{M}_{A(D)} = \bar{D}$.

Example

Let $K \subset \mathbb{C}^n$ compact. Consider

$$P(K) = \{f \in C(K) : f \text{ is a uniform limit of polynomials in } z_1, \ldots, z_n\}$$

which is a uniform algebra with $\mathfrak{M}_{P(K)} = \hat{K}$, the **polynomial hull** of K,

$$\hat{K} = \{z \in \mathbb{C}^n : |p(z)| \leq \|p\|_K, \forall p \in \mathbb{C}[z_1, \ldots, z_n]\}.$$

For $n = 1$, $\hat{K} = K \cup$ bounded components of $\mathbb{C} \setminus K$.
Note that $P(\partial D) \cong P(\bar{D}) = A(D)$.

Example

Let $K \subset \mathbb{C}^n$ compact. Consider

$$P(K) = \{f \in C(K) : f \text{ is a uniform limit of polynomials in } z_1, \ldots, z_n\}$$

which is a uniform algebra with $\mathfrak{M}_{P(K)} = \hat{K}$, the **polynomial hull** of K,

$$\hat{K} = \{z \in \mathbb{C}^n : |p(z)| \leq \|p\|_K, \forall p \in \mathbb{C}[z_1, \ldots, z_n]\}.$$

For $n = 1$, $\hat{K} = K \cup$ bounded components of $\mathbb{C} \setminus K$.
Note that $P(\partial D) \cong P(\bar{D}) = A(D)$.

Example

Let $K \subset \mathbb{C}^n$ compact. Consider

$$P(K) = \{f \in C(K) : f \text{ is a uniform limit of polynomials in } z_1, \ldots, z_n\}$$

which is a uniform algebra with $\mathfrak{M}_{P(K)} = \hat{K}$, the **polynomial hull** of K,

$$\hat{K} = \{z \in \mathbb{C}^n : |p(z)| \leq \|p\|_K, \forall p \in \mathbb{C}[z_1, \ldots, z_n]\}.$$

For $n = 1$, $\hat{K} = K \cup$ bounded components of $\mathbb{C} \setminus K$.
Note that $P(\partial D) \cong P(\bar{D}) = A(D)$.

Example

For $K \subset \mathbb{C}^n$,

$R(K) = \{f \in C(K) : f$ is a uniform limit of rational functions with poles off $K\}$

is a uniform algebra with $\mathfrak{M}_{R(K)} = h_r(K)$, the **rational hull** of K,

$$h_r(K) = \{z \in \mathbb{C}^n : |f(z)| \leq \|f\|_K, \forall \text{ rational function } f \text{ with poles off } K\}$$

For every $K \subset \mathbb{C}$, $h_r(K) = K$.

Example

For $K \subset \mathbb{C}^n$,

$R(K) = \{f \in C(K) : f$ is a uniform limit of rational functions with poles off $K\}$

is a uniform algebra with $\mathfrak{M}_{R(K)} = h_r(K)$, the **rational hull** of K,

$$h_r(K) = \{z \in \mathbb{C}^n : |f(z)| \leq \|f\|_K, \forall \text{ rational function } f \text{ with poles off } K\}$$

For every $K \subset \mathbb{C}$, $h_r(K) = K$.

Definition

If A is a uniform algebra, an **analytic disc** in $\mathfrak{M}_A$ is an $1-1$ continuous map $\sigma : D \to \mathfrak{M}_A$ so that $f \circ \sigma : D \to \mathbb{C}$ is holomorphic, for all $f \in A$.

Conjecture

If $\mathfrak{M}_A \neq X$ there must be an analytic disc in $\mathfrak{M}_A$.

Stolzenberg 1963

There exists $X \subset \mathbb{C}^2$ compact so that $\hat{X} \setminus X \neq \emptyset$, but $\hat{X}$ contains no analytic disc.

Wermer 1970

There exists $X \subset \mathbb{C}^2$ compact so that $h_r(X) \setminus X \neq \emptyset$ but $h_r(X)$ contains no analytic disc.

Definition

If A is a uniform algebra, an **analytic disc** in $\mathfrak{M}_A$ is an $1-1$ continuous map $\sigma : D \to \mathfrak{M}_A$ so that $f \circ \sigma : D \to \mathbb{C}$ is holomorphic, for all $f \in A$.

Conjecture

If $\mathfrak{M}_A \neq X$ there must be an analytic disc in $\mathfrak{M}_A$.

Stolzenberg 1963

There exists $X \subset \mathbb{C}^2$ compact so that $\hat{X} \setminus X \neq \emptyset$, but $\hat{X}$ contains no analytic disc.

Wermer 1970

There exists $X \subset \mathbb{C}^2$ compact so that $h_r(X) \setminus X \neq \emptyset$ but $h_r(X)$ contains no analytic disc.

Definition

If A is a uniform algebra, an **analytic disc** in $\mathfrak{M}_A$ is an $1-1$ continuous map $\sigma : D \to \mathfrak{M}_A$ so that $f \circ \sigma : D \to \mathbb{C}$ is holomorphic, for all $f \in A$.

Conjecture

If $\mathfrak{M}_A \neq X$ there must be an analytic disc in $\mathfrak{M}_A$.

Stolzenberg 1963

There exists $X \subset \mathbb{C}^2$ compact so that $\hat{X} \setminus X \neq \emptyset$, but $\hat{X}$ contains no analytic disc.

Wermer 1970

There exists $X \subset \mathbb{C}^2$ compact so that $h_r(X) \setminus X \neq \emptyset$ but $h_r(X)$ contains no analytic disc.

Definition

If A is a uniform algebra, an **analytic disc** in $\mathfrak{M}_A$ is an $1-1$ continuous map $\sigma : D \to \mathfrak{M}_A$ so that $f \circ \sigma : D \to \mathbb{C}$ is holomorphic, for all $f \in A$.

Conjecture

If $\mathfrak{M}_A \neq X$ there must be an analytic disc in $\mathfrak{M}_A$.

Stolzenberg 1963

There exists $X \subset \mathbb{C}^2$ compact so that $\hat{X} \setminus X \neq \emptyset$, but $\hat{X}$ contains no analytic disc.

Wermer 1970

There exists $X \subset \mathbb{C}^2$ compact so that $h_r(X) \setminus X \neq \emptyset$ but $h_r(X)$ contains no analytic disc.

If $\mathfrak{M}_A \neq X$ is there a weaker form of analyticity in $\mathfrak{M}_A$?

Gleason, 1957

For $\phi, \psi \in \mathfrak{M}_A$ define

$$\phi \sim \psi \text{ iff } \|\phi - \psi\| = \sup\{|\phi(f) - \psi(f)| : f \in A, \|f\| \leq 1\} < 2.$$

Then $\sim$ is an equivalence relationship on $\mathfrak{M}_A$ and the equivalence classes of $\sim$ are called **Gleason Parts**.

Disc Algebra

If $|s|, |t| < 1, f \in A(D)$ and $\|f\| \leq 1$ then by Schwarz's Lemma

$$|f(s) - f(t)| \leq \left|\frac{s-t}{1-\bar{s}t}\right| |1 - \bar{f(s)}f(t)|$$

so $s \sim t$.

If $|s| = 1$ and $|t| \leq 1$ using automorphisms of D we may show $s \nsim t$.

Gleason parts for $A(D) : D, \{z\}, z \in \partial D$.

Example

For X a compact T_2 space, $C(X)$ has only trivial Gleason parts.

If $\mathfrak{M}_A \neq X$ is there a weaker form of analyticity in $\mathfrak{M}_A$?

Gleason, 1957

For $\phi, \psi \in \mathfrak{M}_A$ define

$$\phi \sim \psi \text{ iff } \|\phi - \psi\| = \sup\{|\phi(f) - \psi(f)| : f \in A, \|f\| \leq 1\} < 2.$$

Then $\sim$ is an equivalence relationship on $\mathfrak{M}_A$ and the equivalence classes of $\sim$ are called **Gleason Parts**.

Disc Algebra

If $|s|, |t| < 1, f \in A(D)$ and $\|f\| \leq 1$ then by Schwarz's Lemma

$$|f(s) - f(t)| \leq \left|\frac{s-t}{1-\bar{s}t}\right| |1 - f(\bar{s})f(t)|$$

so $s \sim t$.
If $|s| = 1$ and $|t| \leq 1$ using automorphisms of D we may show $s \nsim t$.
Gleason parts for $A(D) : D, \{z\}, z \in \partial D$.

Example

For X a compact T_2 space, $C(X)$ has only trivial Gleason parts.

If $\mathfrak{M}_A \neq X$ is there a weaker form of analyticity in $\mathfrak{M}_A$?

Gleason, 1957

For $\phi, \psi \in \mathfrak{M}_A$ define

$$\phi \sim \psi \text{ iff } \|\phi - \psi\| = \sup\{|\phi(f) - \psi(f)| : f \in A, \|f\| \leq 1\} < 2.$$

Then $\sim$ is an equivalence relationship on $\mathfrak{M}_A$ and the equivalence classes of $\sim$ are called **Gleason Parts**.

Disc Algebra

If $|s|, |t| < 1, f \in A(D)$ and $\|f\| \leq 1$ then by Schwarz's Lemma

$$|f(s) - f(t)| \leq \left| \frac{s-t}{1-\bar{s}t} \right| |1 - f(\bar{s})f(t)|$$

so $s \sim t$.

If $|s| = 1$ and $|t| \leq 1$ using automorphisms of D we may show $s \nsim t$.

Gleason parts for $A(D) : D, \{z\}, z \in \partial D$.

Example

For X a compact T_2 space, $C(X)$ has only trivial Gleason parts.

If $\mathfrak{M}_A \neq X$ is there a weaker form of analyticity in $\mathfrak{M}_A$?

Gleason, 1957

For $\phi, \psi \in \mathfrak{M}_A$ define

$$\phi \sim \psi \text{ iff } \|\phi - \psi\| = \sup\{|\phi(f) - \psi(f)| : f \in A, \|f\| \leq 1\} < 2.$$

Then $\sim$ is an equivalence relationship on $\mathfrak{M}_A$ and the equivalence classes of $\sim$ are called **Gleason Parts**.

Disc Algebra

If $|s|, |t| < 1, f \in A(D)$ and $\|f\| \leq 1$ then by Schwarz's Lemma

$$|f(s) - f(t)| \leq \left| \frac{s-t}{1-\bar{s}t} \right| |1 - f(\bar{s})f(t)|$$

so $s \sim t$.
If $|s| = 1$ and $|t| \leq 1$ using automorphisms of D we may show $s \nsim t$.
Gleason parts for $A(D) : D, \{z\}, z \in \partial D$.

Example

For X a compact T_2 space, $C(X)$ has only trivial Gleason parts.

If $\mathfrak{M}_A \neq X$ is there a weaker form of analyticity in $\mathfrak{M}_A$?

Gleason, 1957

For $\phi, \psi \in \mathfrak{M}_A$ define

$$\phi \sim \psi \text{ iff } \|\phi - \psi\| = \sup\{|\phi(f) - \psi(f)| : f \in A, \|f\| \leq 1\} < 2.$$

Then $\sim$ is an equivalence relationship on $\mathfrak{M}_A$ and the equivalence classes of $\sim$ are called **Gleason Parts**.

Disc Algebra

If $|s|, |t| < 1, f \in A(D)$ and $\|f\| \leq 1$ then by Schwarz's Lemma

$$|f(s) - f(t)| \leq \left|\frac{s-t}{1-\bar{s}t}\right| |1 - \bar{f(s)}f(t)|$$

so $s \sim t$.
If $|s| = 1$ and $|t| \leq 1$ using automorphisms of D we may show $s \nsim t$.
Gleason parts for $A(D) : D, \{z\}, z \in \partial D$.

Example

For X a compact T_2 space, $C(X)$ has only trivial Gleason parts.

If $\mathfrak{M}_A \neq X$ is there a weaker form of analyticity in $\mathfrak{M}_A$?

Gleason, 1957

For $\phi, \psi \in \mathfrak{M}_A$ define

$$\phi \sim \psi \text{ iff } \|\phi - \psi\| = \sup\{|\phi(f) - \psi(f)| : f \in A, \|f\| \leq 1\} < 2.$$

Then $\sim$ is an equivalence relationship on $\mathfrak{M}_A$ and the equivalence classes of $\sim$ are called **Gleason Parts**.

Disc Algebra

If $|s|, |t| < 1, f \in A(D)$ and $\|f\| \leq 1$ then by Schwarz's Lemma

$$|f(s) - f(t)| \leq \left| \frac{s-t}{1-\bar{s}t} \right| |1 - f(\bar{s})f(t)|$$

so $s \sim t$.
If $|s| = 1$ and $|t| \leq 1$ using automorphisms of D we may show $s \nsim t$.
Gleason parts for $A(D) : D, \{z\}, z \in \partial D$.

Example

For X a compact T_2 space, $C(X)$ has only trivial Gleason parts.

Wermer, Hoffman

If A is a Dirichlet or logmodular algebra (e.x. $A(D)$, $P(X)$ for $X \subset \mathbb{C}$ compact with $\mathbb{C} \setminus X$ connected, $H^\infty(D)$), then every Gleason part for A is either trivial or an analytic disc.

Conjecture

If the uniform algebra A on X has only trivial Gleason parts then $A = C(X)$.

Cole, 1968

There exists A on X such that $A \neq C(X)$ but it has only trivial Gleason parts.

Cole, Ghosh, Izzo, 2000

There is $X \subset \mathbb{C}^3$ compact such that $\hat{X} \setminus X \neq \emptyset$ but $P(X)$ has only trivial Gleason parts.

Conjecture

If the uniform algebra A on X has only trivial Gleason parts then $A = C(X)$.

Cole, 1968

There exists A on X such that $A \neq C(X)$ but it has only trivial Gleason parts.

Cole, Ghosh, Izzo, 2000

There is $X \subset \mathbb{C}^3$ compact such that $\hat{X} \setminus X \neq \emptyset$ but $P(X)$ has only trivial Gleason parts.

Conjecture

If the uniform algebra A on X has only trivial Gleason parts then $A = C(X)$.

Cole, 1968

There exists A on X such that $A \neq C(X)$ but it has only trivial Gleason parts.

Cole, Ghosh, Izzo, 2000

There is $X \subset \mathbb{C}^3$ compact such that $\hat{X} \setminus X \neq \emptyset$ but $P(X)$ has only trivial Gleason parts.

Gamelin, Rossi

There are not known examples of disconnected Gleason parts but there existence seems very likely.

Question

What spaces can occur as Gleason parts for a uniform algebra?

If P is a Gleason part for A it is completely regular (since $P \subset \mathfrak{M}_A$).
P is also σ-compact since if $\phi \in P$, $P = \bigcup_{n=1}^{\infty} \{\psi \in \mathfrak{M}_A : \|\phi - \psi\| \leq 2 - 1/n\}$.

Garnett, 1967

If P is a completely regular, σ-compact space there is uniform algebra A such that $\mathfrak{M}_A$ contains a homeomorphic image of P which is a Gleason part for A, and such that $A|_P = C_b(P)$ (the space of continuous and bounded functions on P).

So nontrivial Gleason parts carry no analytic structure in general.
However in Garnett's construction $\mathfrak{M}_A$ contains many analytic discs.

Gamelin, Rossi

There are not known examples of disconnected Gleason parts but there existence seems very likely.

Question

What spaces can occur as Gleason parts for a uniform algebra?

If P is a Gleason part for A it is completely regular (since $P \subset \mathfrak{M}_A$).
P is also σ-compact since if $\phi \in P, P = \bigcup_{n=1}^{\infty} \{\psi \in \mathfrak{M}_A : \|\phi - \psi\| \leq 2 - 1/n\}$.

Garnett, 1967

If P is a completely regular, σ-compact space there is uniform algebra A such that $\mathfrak{M}_A$ contains a homeomorphic image of P which is a Gleason part for A, and such that $A|_P = C_b(P)$ (the space of continuous and bounded functions on P).

So nontrivial Gleason parts carry no analytic structure in general.
However in Garnett's construction $\mathfrak{M}_A$ contains many analytic discs.

Gamelin, Rossi

There are not known examples of disconnected Gleason parts but there existence seems very likely.

Question

What spaces can occur as Gleason parts for a uniform algebra?

If P is a Gleason part for A it is completely regular (since $P \subset \mathfrak{M}_A$).

P is also σ-compact since if $\phi \in P$, $P = \bigcup_{n=1}^{\infty} \{\psi \in \mathfrak{M}_A : \|\phi - \psi\| \leq 2 - 1/n\}$.

Garnett, 1967

If P is a completely regular, σ-compact space there is uniform algebra A such that $\mathfrak{M}_A$ contains a homeomorphic image of P which is a Gleason part for A, and such that $A|_P = C_b(P)$ (the space of continuous and bounded functions on P).

So nontrivial Gleason parts carry no analytic structure in general.
However in Garnett's construction $\mathfrak{M}_A$ contains many analytic discs.

Gamelin, Rossi

There are not known examples of disconnected Gleason parts but there existence seems very likely.

Question

What spaces can occur as Gleason parts for a uniform algebra?

If P is a Gleason part for A it is completely regular (since $P \subset \mathfrak{M}_A$).
P is also σ-compact since if $\phi \in P$, $P = \bigcup_{n=1}^{\infty} \{\psi \in \mathfrak{M}_A : \|\phi - \psi\| \leq 2 - 1/n\}$.

Garnett, 1967

If P is a completely regular, σ-compact space there is uniform algebra A such that $\mathfrak{M}_A$ contains a homeomorphic image of P which is a Gleason part for A, and such that $A|_P = C_b(P)$ (the space of continuous and bounded functions on P).

So nontrivial Gleason parts carry no analytic structure in general.
However in Garnett's construction $\mathfrak{M}_A$ contains many analytic discs.

Gamelin, Rossi

There are not known examples of disconnected Gleason parts but there existence seems very likely.

Question

What spaces can occur as Gleason parts for a uniform algebra?

If P is a Gleason part for A it is completely regular (since $P \subset \mathfrak{M}_A$).
P is also σ-compact since if $\phi \in P$, $P = \bigcup_{n=1}^{\infty} \{\psi \in \mathfrak{M}_A : \|\phi - \psi\| \leq 2 - 1/n\}$.

Garnett, 1967

If P is a completely regular, σ-compact space there is uniform algebra A such that $\mathfrak{M}_A$ contains a homeomorphic image of P which is a Gleason part for A, and such that $A|_P = C_b(P)$ (the space of continuous and bounded functions on P).

So nontrivial Gleason parts carry no analytic structure in general.
However in Garnett's construction $\mathfrak{M}_A$ contains many analytic discs.

Gamelin, Rossi

There are not known examples of disconnected Gleason parts but there existence seems very likely.

Question

What spaces can occur as Gleason parts for a uniform algebra?

If P is a Gleason part for A it is completely regular (since $P \subset \mathfrak{M}_A$).
P is also σ-compact since if $\phi \in P$, $P = \bigcup_{n=1}^{\infty}\{\psi \in \mathfrak{M}_A : \|\phi - \psi\| \leq 2 - 1/n\}$.

Garnett, 1967

If P is a completely regular, σ-compact space there is uniform algebra A such that $\mathfrak{M}_A$ contains a homeomorphic image of P which is a Gleason part for A, and such that $A|_P = C_b(P)$ (the space of continuous and bounded functions on P).

So nontrivial Gleason parts carry no analytic structure in general.
However in Garnett's construction $\mathfrak{M}_A$ contains many analytic discs.

Gamelin, Rossi

There are not known examples of disconnected Gleason parts but there existence seems very likely.

Question

What spaces can occur as Gleason parts for a uniform algebra?

If P is a Gleason part for A it is completely regular (since $P \subset \mathfrak{M}_A$).
P is also σ-compact since if $\phi \in P, P = \bigcup_{n=1}^{\infty} \{\psi \in \mathfrak{M}_A : \|\phi - \psi\| \leq 2 - 1/n\}$.

Garnett, 1967

If P is a completely regular, σ-compact space there is uniform algebra A such that $\mathfrak{M}_A$ contains a homeomorphic image of P which is a Gleason part for A, and such that $A|_P = C_b(P)$ (the space of continuous and bounded functions on P).

So nontrivial Gleason parts carry no analytic structure in general.
However in Garnett's construction $\mathfrak{M}_A$ contains many analytic discs.

Question

Which spaces occur as Gleason parts for a uniform algebra without analytic discs?

Izzo, P.

If P is a completely regular, σ-compact space, there is a uniform algebra A on a compact T_2 space X without analytic discs, but having a part homeomorphic to P lying in $\mathfrak{M}_A \setminus X$. Furthermore, $A|_P = C_b(P)$.

Question

Which spaces occur as Gleason parts for a uniform algebra without analytic discs?

Izzo, P.

If P is a completely regular, σ-compact space, there is a uniform algebra A on a compact T_2 space X without analytic discs, but having a part homeomorphic to P lying in $\mathfrak{M}_A \setminus X$. Furthermore, $A|_P = C_b(P)$.

Question

Which spaces occur as a Gleason part for a uniform algebra without analytic discs over a metrizable space?

Izzo, P.

Let P be a σ-compact, metrizable space. There exists a uniform algebra A (necessarily separable) on a metrizable, compact space X, without analytic discs, and such that $\mathfrak{M}_A \setminus X$ contains a homeomorphic image of P constituting a Gleason part. Furthermore, $A|_P = C(X)|_P$.

Question

Which spaces occur as a Gleason part for a uniform algebra without analytic discs over a metrizable space?

Izzo, P.

Let P be a σ-compact, metrizable space. There exists a uniform algebra A (necessarily separable) on a metrizable, compact space X, without analytic discs, and such that $\mathfrak{M}_A \setminus X$ contains a homeomorphic image of P constituting a Gleason part. Furthermore, $A|_P = C(X)|_P$.

Question

Which spaces can occur as a Gleason part for a finitely generated uniform algebra? Equivalently for a $P(X)$, $X \subset \mathbb{C}^n$ compact?

Izzo, P.

If P is a locally compact space of finite topological dimension, there exists an $X \subset \mathbb{C}^n$ for some $n \in \mathbb{N}$ compact, such that $\hat{X}$ contains no analytic discs, but does contain a part homeomorphic to P lying in $\hat{X} \setminus X$.

Open question

Can $\mathbb{Q}$ occur as a Gleason part for a finitely generated uniform algebra?

Question

Which spaces can occur as a Gleason part for a finitely generated uniform algebra? Equivalently for a $P(X)$, $X \subset \mathbb{C}^n$ compact?

Izzo, P.

If P is a locally compact space of finite topological dimension, there exists an $X \subset \mathbb{C}^n$ for some $n \in \mathbb{N}$ compact, such that $\hat{X}$ contains no analytic discs, but does contain a part homeomorphic to P lying in $\hat{X} \setminus X$.

Open question

Can $\mathbb{Q}$ occur as a Gleason part for a finitely generated uniform algebra?

Question

Which spaces can occur as a Gleason part for a finitely generated uniform algebra? Equivalently for a $P(X)$, $X \subset \mathbb{C}^n$ compact?

Izzo, P.

If P is a locally compact space of finite topological dimension, there exists an $X \subset \mathbb{C}^n$ for some $n \in \mathbb{N}$ compact, such that $\hat{X}$ contains no analytic discs, but does contain a part homeomorphic to P lying in $\hat{X} \setminus X$.

Open question

Can $\mathbb{Q}$ occur as a Gleason part for a finitely generated uniform algebra?

THANK YOU!