Recent results on polynomial inequalities

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Original A. A. Markov inequality (1889):
Let P be a real polynomial, $\deg(P) = n$. Then

$$\|P'\|_I \leq n^2 \|P\|_I$$

where $\|.\|_I$ is the sup norm over $I = [-1, 1]$.

Later, Turán asked about reverse type inequality, under certain natural assumptions.
Let P be a real polynomial, $\deg(P) = n$. Assume that all the zeros of P belong to I. Then

$$\|P'\|_I \geq \frac{\sqrt{n}}{6} \|P\|_I.$$

(Turán, 1939)

On the unit disk $\mathbb{D} = \{|z| < 1\}$ the corresponding inequalities are:

(Bernstein/M. Riesz, 1914) Let P be a complex polynomial, $\deg(P) = n$. Then

$$\|P'\|_{\mathbb{D}} \leq n\|P\|_{\mathbb{D}}$$

where $\|.\|_{\mathbb{D}}$ is the sup norm over the closed unit disk.

(Turán, 1939) Let P be a complex polynomial, $\deg(P) = n$. Assume that all zeros of P belong to the closed unit disk. Then

$$\|P'\|_{\mathbb{D}} \geq \frac{n}{2}\|P\|_{\mathbb{D}}.$$

Actually, a bit stronger assertion was proved: if $|z| = 1$ is such that $|P(z)| = \|P\|_{\mathbb{D}}$, then

$$|P'(z)| \geq \frac{n}{2}\|P\|_{\mathbb{D}}.$$

Turán's inequality on $\mathbb{D}$ was soon generalized to ellipses by Erőd (1939). Let $0 \le a \le 1$, $E := \{x + iy : x^2 + y^2/a^2 \le 1\}$. Assume that P is a complex polynomial with all zeros in E, $\deg(P) = n$. Then, for any $z \in \partial E$,

$$|P'(z)| \ge \frac{n}{2} \frac{a}{\sqrt{1 + a^2 - |z|^2}} |P(z)|.$$

Note that $a/\sqrt{1 + a^2 - |z|^2} \ge a$, hence

$$\|P'\|_E \ge \frac{n}{2} a \|P\|_E.$$

For general sets, a Turán type inequality was established by Levenberg and Poletsky (2002): Let $K \subset \mathbf{C}$ be a convex compact set. Denote the diameter of K by $\mathrm{diam}(K)$. Assume that P is a complex polynomial with all zeros in K, $\deg(P) = n$. Then

$$\|P'\|_K \geq \frac{1}{20} \frac{1}{\mathrm{diam}(K)} \sqrt{n} \|P\|_K.$$

Later, Révész (2006) established a Turán type inequality for the same class of sets: Additionally, denote the width of K by $w(K)$. Then

$$\|P'\|_K \geq 0.0003 \frac{w(k)}{\mathrm{diam}(K)^2} n \|P\|_K.$$

Asymptotically sharp Bernstein type inequality was established for a general class of sets (N-Totik, 2006): Let $K \subset \mathbf{C}$ be a compact set such that ∂K consists of finitely many disjoint C^2 smooth Jordan curves (each lying exterior of the others). Then for any $z \in \partial K$ and polynomial P, $\deg(P) = n$, we have

$$|P'(z)| \le (1 + o(1))n \frac{\partial}{\partial \mathbf{n}(z)} g_K(z) \|P\|_K$$

where $\partial/\partial\mathbf{n}(z) g_K(z)$ is the normal derivative of Green's function of K, $g_K(z) = g_{\mathbf{C}_\infty \setminus K}(z, \infty)$ and $o(1)$ denotes an error term that depends on n,K and z and is independent of P and tends to 0 as $n \to \infty$.

This is sharp: $o(1)$ cannot be removed and $\frac{\partial}{\partial \mathbf{n}(z)}$ cannot be replaced with smaller const. $\rightsquigarrow$ potential theory

Our work in progress: Let $K \subset \mathbf{C}$ be a compact set such that ∂K is an analytic Jordan curve. Denote Green's function of K by $g_K(z) = g_{\mathbf{C}_\infty \setminus K}(z, \infty)$. Assume that P is a complex polynomial with all zeros in K, $\deg(P) = n$. Let $z_0 \in \partial K$ be such that $|P(z_0)| = \|P\|_K$. Conjecture:

$$|P'(z_0)| \geq (1 - o(1)) \frac{\partial}{\partial \mathbf{n}(z_0)} g_K(z_0) \frac{n}{2} \|P\|_K$$

where $o(1)$ depends on n, K and z_0 but it is independent of P and tends to 0 as $n \to \infty$.
Once it is verified, we immediately have a stronger (error-term free) version:

$$\|P'\|_K \geq \omega_0 \frac{n}{2} \|P\|_K$$

where $\omega_0 = \min\{\partial/\partial \mathbf{n}(z) g_K(z) : \ z \in \partial K\}$.

Some tools used/related: zero free approximation (e.g. Gauthier, Danielyan, Khruschev), approximation of holomorphic functions with simple partial fractions $\sum_k 1/(z - z_k)$ (e.g. Dolzhenko, Danchenko)

A (open?) question arised during research: Let γ be a C^2 smooth Jordan curve. Let $z_1, \dots, z_n$ be different points, $z_k \in \mathrm{Int}\gamma$, and let $w_{k,j} \in \mathbf{C}$, $k = 1, \dots, n$, $j = 0, 1, \dots, m_k$ be given with $w_{k,0} \neq 0$ (for all $k = 1, \dots, n$; as data for Hermite interpolation). Does there exist a polynomial $P(z) = c(z - z_1) \dots (z - z_N)$ such that $z_1, \dots, z_N \in \gamma$ and $P^{(j)}(z_k) = w_{k,j}$ for all k and j?
$\sim$ approximation with polynomials with prescribed/restricted zeros; taking log derivative, approximation with simple partial fractions (spf)/simplest fractions/logarithmic derivatives of polynomials (ldp).
Note that N may depend on the values $w_{j,k}$ too. It is interesting even when $\gamma = \partial\mathbb{D}$ and $m_1 = \dots = m_k = 0$.

Comparing Erőd's result with our conjecture:
As earlier, let $0 \leq a \leq 1$, $E := \{x + iy : \; x^2 + y^2/a^2 \leq 1\}$ and denote Green's function of E by $g_E(z) = g_{\mathbf{C}_\infty \setminus E}(z, \infty)$. Then, we have at $z \in \partial E$

$$\frac{\partial}{\partial \mathbf{n}(z)} g_E(z) \geq \frac{a}{\sqrt{1 + a^2 - |z|^2}}.$$

Comparing Révész' result with our conjecture:
Conjecture: for any convex set K with C^2 smooth boundary,

$$\frac{\partial}{\partial \mathbf{n}(z)} g_K(z) \geq \frac{1}{2\pi \mathrm{diam}(K)}$$

at any $z \in \partial K$.

Assuming this, and using the immediate assertion

$$\frac{1}{2\pi \mathrm{diam}(K)} \geq 0.0003 \frac{w(k)}{\mathrm{diam}(K)^2} \frac{1}{2}.$$

we can derive Révész' result.

It is worth mentioning two new papers:
* Glazyrina-Révész, arxiv 1805.04822, compact convex sets, L^q norms, Turán type inequality for polynomials
* Erdélyi, manuscript, sharp estimates for real polynomials on $[0, 1]$ when there are fixed number of zeros at 0.

All comments, suggestions are welcome.

Thank you for your attention!