Wandering domains and (post)-singular values

Xavier Jarque i Ribera

Universitat de Barcelona & Barcelona Graduate School of Mathematics

UNIVERSITAT DE BARCELONA

BGSMath
BARCELONA GRADUATE SCHOOL OF MATHEMATICS

New developments in complex analysis and function theory

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Introduction

Let f be a rational $f : \hat{\mathbb{C}} \mapsto \hat{\mathbb{C}}$, or transcendental $f : \mathbb{C} \mapsto \{\mathbb{C}, \hat{\mathbb{C}}\}$ map. Consider the dynamical system defined by the iterates of f, that is $\{f^n(z_0)\}_{n\geq 0},\ z_0 \in \{\hat{\mathbb{C}}, \mathbb{C}\}$ (if defined). We divide the phase space in two completely invariant subsets:

(a) The Fatou set: $z \in \hat{\mathbb{C}}$ is in the Fatou set if f is normal at z. That is if there exists a neighborhood U of z such that $\{f^n|_U\}_{n\geq 0}$ converges locally uniformly to a holomorphic map ψ, or to infinity (limit function). We denote the Fatou set by $\mathcal{F}(f)$.

(b) The Julia set: The complement of $\mathcal{F}(f)$ in $\hat{\mathbb{C}}$. We denote it by $\mathcal{J}(f)$.

Remark. The set $\mathcal{F}(f)$ is open and the set $\mathcal{J}(f)$ is closed (and non empty). Each connected component of $\mathcal{F}(f)$ is called a Fatou domain or Fatou component. Fatou domains are mapped into Fatou domains.

Non (eventually) periodic Fatou domains

Definition. Accordingly if U is a Fatou component it might be either eventually periodic, or non. If U is not eventually periodic, we say that U is a wandering domain (of f). In this case we have

$$f^n(U) \cap f^m(U) = \emptyset \quad \forall n \neq m, \quad n, m \in \mathbb{Z}.$$

Theorem (Sullivan 1985): Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map and let U be a Fatou domain of f. Then U is eventually periodic.
In other words, rational functions do not have wandering domains.

Remark. We restrict our attention to transcendental functions.

The post-singular set

Let f be a transcendental entire map. We denote by $\mathcal{S}(f)$ the set of (finite) singularities of f^{-1} (critical values, asymptotic values or accumulations of those values).

Theorem. Let z_0 be an attracting fixed point of f and let

$$A(z_0) = \{z \in \mathbb{C} \mid f^n(z) \to z_0, \ n \to \infty\}$$

be its (open) basin of attraction. We denote by $A^\star(z_0) \subset A(z_0)$ the connected component where z_0 belongs to (immediate basin of attraction). Then, there exists $s \in \mathcal{S}(f)$ such that $s \in A^\star(z_0)$.

Definition. The post-singular set of f is defined as follows

$$P := \mathcal{P}(f) = \bigcup_{s \in \mathcal{S}(f)} \bigcup_{n \geq 0} f^n(s).$$

Classes of transcendental entire maps

Definition. We say that $f \in \mathcal{S}$ (Speiser class) if $\mathcal{S}(f)$ is finite. We say that $f \in \mathcal{B}$ (Eremenko-Lyubich class) if $\mathcal{S}(f)$ is bounded.

Theorem (Eremenko-Lyubich, Golberg-Keen 1986): If $f \in \mathcal{S}$ then f has no wandering domains.

Theorem (Bishop, 2015): There is $f \in \mathcal{B}$ having two symmetric (grand orbits of) (non univalent) wandering domains.

Remark. Later K. Lazebnik proved that those wanderings are are bounded Fatou domains (in Fagella-J.-Lazebnik the example is modified to get a univalent one).

Remark. Today afternoon D. Martí-Pete will present an alternative construction to Bishop's example for wandering domains in class $\mathcal{B}$.

Examples of wandering domains I

Theorem (Baker and Töpfer): If $U \subset \mathcal{F}(f)$ is multiply connected then

(a) $f^n|_U \mapsto \infty$ (uniformly on compact subset of U),

(b) U is bounded, and

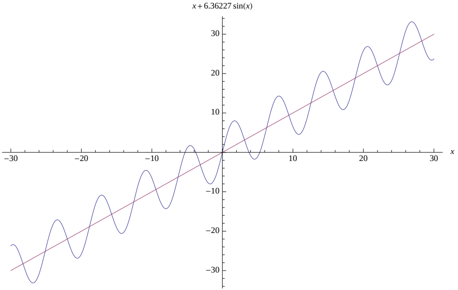
(c) U is a wandering domain.

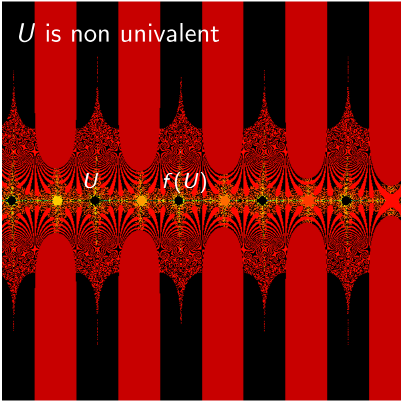
Theorem (Baker's example, 70's): Let $g(z) = \frac{1}{4e} z^2 \prod_{n=1}^{\infty} \left(1 + \frac{z}{a_n}\right)$. If the sequence $\{a_j \in \mathbb{R}^+\}_{j \geq 0}$ is appropriately chosen then g has a (Baker)-wandering (multiply connected, non univalent) domain.

We refer to Bergweiler-Rippon-Stallard or Kisaka-Shishikura for multiply connected wandering domains.

Examples of wandering domains II

Let $f(z) = z + \lambda_0 \sin(z)$ with $\lambda_0 \approx 6.36227$.





Examples of wandering domains III: Herman-Sullivan, 80'

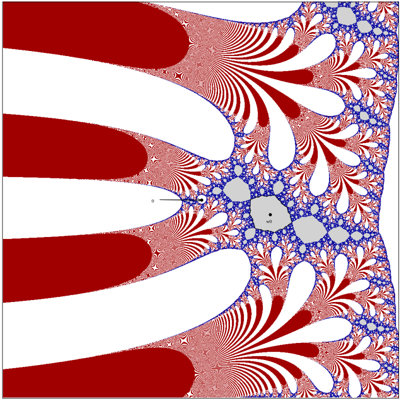
$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{z-1+e^{-z}+2\pi i} & \mathbb{C} \\ {\scriptstyle e^{-z}}\downarrow & & \downarrow{\scriptstyle e^{-z}} \\ \mathbb{C}\setminus\{0\} & \xrightarrow{h(w)=we^{-w+1}} & \mathbb{C}\setminus\{0\} \end{array}$$

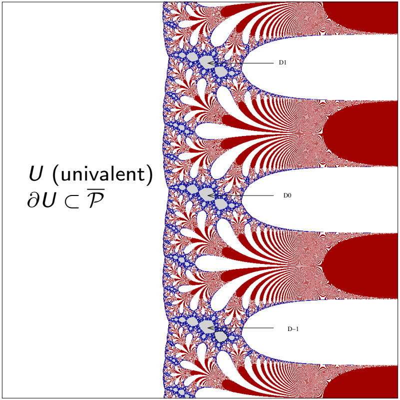
Lemma: Assume $f \circ g = g \circ f$ (they are permutable) and $f = g + c$ for some $c \in \mathbb{C}$. Then $J(f) = J(g)$.

Proposition: Let $f(z) = z - 1 + e^{-z}$. The function $g(z) = f(z) + 2\pi i$ has a wandering domain.

Proof of the Proposition: $z_n = 2n\pi i,\ n \in \mathbb{Z}$ are superattracting fixed points for f (the lifts of the superattracting fixed point $w = 1$ for h). So, since $\mathcal{J}(g) = \mathcal{J}(f)$ and $g(z_n) = z_{n+1}$ the basins of attraction become non univalent wandering components.

Examples of wandering domains III: Lift argument





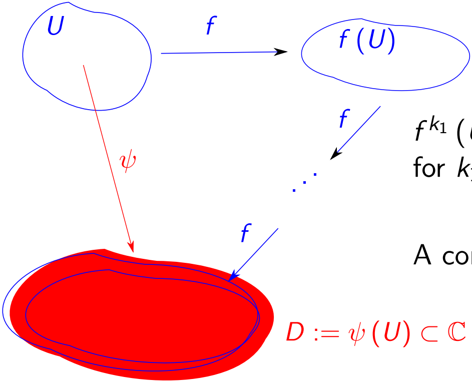
$$h(w) = c_1(\lambda)\, w^2 \exp(-w) \qquad f(z) = c_2(\lambda) + 2z - \exp(z)$$

$$\lambda = \exp\left(\pi i \left(1 - \sqrt{5}\right)\right)$$

Constant limit functions

Theorem (Fatou 1920): Let U a wandering domain of f. All limit functions of the (convergence) sequences $\{f^{n_k}|_U\}$ are constant.

Idea of the proof.



$f^{k_1}(U) \cap f k_2(U) \neq \emptyset$,
for k_1, k_2 large enough.

A contradiction.

Dynamical classification of wandering domains

Theorem (Fatou 1920). Let U a wandering domain of f. All limit functions of the (convergence) sequences $\{f^{n_k}|_U\}$ are constant.

- $\{f^n|_U\} \to \infty$ (escaping)
- $\{f^{n_k}|_U\} \to \infty$ and $\{f^{m_k}|_U\} \to a \in \mathcal{J}(f) \subset \mathbb{C}$ (oscillating)
- If $\{f^{n_k}|_U\} \to a$ then $a \neq \infty$ (bounded) $\leftarrow$ dynamically!!!

Remark. All previous (multiply connected and lift's) examples are escaping.

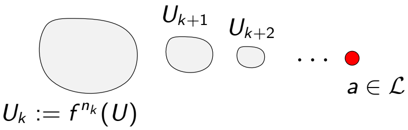
Theorem (Eremenko-Lyubich (1987) and Bishop (2015)). There exists an entire function f which has an oscillating wandering component U (with infinitely many finite constant limit functions). Such f can be chosen in class $\mathcal{B}$.

Remark: There are no examples of the third type.

(Post)-Singular set and wandering domains

Let U be a wandering domain of f.

$$\mathcal{L} = \{a \in \hat{\mathbb{C}} \mid \exists n_k \to \infty \mid f^{n_k}_{|U} \to a\}$$



$$\mathcal{P} = \bigcup_{s \in \mathcal{S}(f)} \bigcup_{n \geq 0} f^n(s)$$

Theorem (Baker, 1976).

$$\mathcal{L} \subset \overline{\mathcal{P}} \cup \infty.$$

Theorem (BHKMT, 1993).

$$\mathcal{L} \subset \mathcal{P}' \cup \infty,$$

where $\mathcal{P}'$ is the set of finite limit points of $\mathcal{P}$.

Corollary (BHKMT). $\mathcal{J}(\exp(z)) = \mathbb{C}$. $(\mathcal{P}' = \emptyset)$.

Wandering domains in class $\mathcal{B}$ and singular values

Theorem (Eremenko-Lyubich 1985). Let $f \in \mathcal{B}$. Then, wandering domains are either oscillating or bounded.

Question. Let $f \in \mathcal{B}$. Assume

$$\lim_{n\to\infty} \inf_{s\in\mathcal{S}(f)} |f^n(s)| = \infty. \tag{1}$$

Can f have a wandering domain? (If any, it would be univalent)

Theorem (Mihaljević-Rempe 2013). Let $f \in \mathcal{B}$ satisfying (1) and condition $(\star)$. Then f has no wandering domains.

Remark. Bishop's example having a wandering domain does not satisfy (1).

A question on wandering domains in class $\mathcal{B}$ and singular values

Question. Let $f \in \mathcal{B}$. Let U a (oscillating or bounded) wandering domain. Should there $m_k \to \infty$ and $s \in \mathcal{S}(f)$ so that $f^{n_k}_{|U} \to s$?

We know there exist $n_k \to \infty$ and $a \in \mathcal{P}' \cap \mathbb{C}$ such that $f^{n_k}_{|U} \to a$.

Wandering domains and singularities of meromorphic maps

Theorem (Baker and Zheng). Let f be a meromorphic transcendental map. Let U a wandering domain.

- Any limit function of iterates in U (i.e., $f^{n_k}|_U$), is a constant which belongs to $\mathcal{P}' \cup \infty$.
- If $f^n|_U \to a \in \hat{\mathbb{C}}$ then $a = \infty \in S(f)'$.

Wandering domains and singularities of meromorphic maps

(joint work with Baranski, Fagella and Karpinska)

Theorem. Let f be a transcendental meromorphic map. Let U be a wandering domain. Denote by U_n the Fatou component such that $f^n(U) \subset U_n$. Then, for every $z \in U$ there exists a sequence $\{p_n \in \mathcal{P}\}_{n \geq 0}$ such that

$$\frac{\text{dist}(p_n, U_n)}{\text{dist}(f^n(z), \partial U_n)} \to 0, \quad \text{as } n \to \infty.$$

$(\text{dist}(\alpha, A) = \inf\{|\alpha - w| \mid w \in A\})$.

In particular, if the diameter of U_n is uniformly bounded, then

$$\text{dist}(p_n, U_n) \to 0 \quad \text{as } n \to \infty.$$

Topologically hyperbolic meromorphic maps

Definition. A meromorphic transcendental function f is called topologically hyperbolic if

$$\mathrm{dist}(\mathcal{P}(f), \mathcal{J}(f) \cap \mathbb{C}) > 0.$$

Remark 1. This is a weaker condition than hyperbolicity ($\overline{\mathcal{P}(f)}$ bounded and disjoint of the Julia set). (Newton's map of entire functions)

Remark 2. Topologically hyperbolic maps cannot have parabolic cycles, or rotation domains.

Remark 3. Topologically hyperbolic maps cannot have oscillating or bounded wandering domains.

Topologically hyperbolic meromorphic maps

Corollary. Let f topologically hyperbolic. Suppose that $U_n \cap P(f) = \emptyset$ for $n > 0$. Fix $z \in U$. Then for every $r > 0$ there exists n_0 such that for every $n \geq n_0$, we have $\mathbb{D}(f^n(z), r) \subset U_n$. In particular, $\operatorname{diam}(U_n) \to \infty$, as $n \to \infty$.

Proof.

- Previous theorem implies

$$\frac{\operatorname{dist}(p_n, U_n)}{\operatorname{dist}(f^n(z), \partial U_n)} < \varepsilon_n, \quad \varepsilon_n \to 0,\ n \to \infty.$$

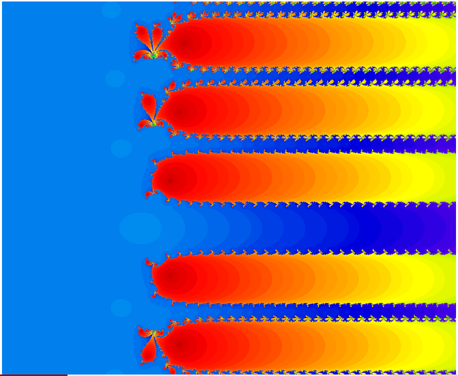
- f topologically hyperbolic and $U_n \cap P(f) = \emptyset$ implies $\operatorname{dist}(p_n, U_n) > c > 0$. Hence $\operatorname{dist}(f^n(z), \partial U_n) \to \infty$ as $n \to \infty$.

Topologically hyperbolic meromorphic maps

Example. The function

$$N_f(z) = \frac{\exp(z)\,(z-1)}{\exp(z)+1},$$

which is the Newton method of $f(z) = \exp(z) + z$ has no wandering domains.



Knossos, Crete, Grece

Thank you for the attention