Dual exponential polynomials and linear differential equations

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Solutions are entire functions

The solutions of the linear differential equation

$$f^{(n)} + a_{n-1}(z)f^{(n-1)} + \cdots + a_1(z)f' + a_0(z)f = 0 \qquad (1)$$

with entire coefficients $a_0(z), \ldots, a_{n-1}(z)$ are entire.

To avoid ambiguity, we assume that $a_0(z) \not\equiv 0$.

Theorems by Wittich and Frei

Wittich's theorem. *The coefficients* $a_0(z), \ldots, a_{n-1}(z)$ *of* (1) *are polynomials if and only if all solutions of* (1) *are of finite order.*

Frei's theorem. *Suppose that at least one coefficient in* (1) *is transcendental, and that* $a_j(z)$ *is the last transcendental coefficient, that is, the coefficients* $a_{j+1}(z), \ldots, a_{n-1}(z)$*, if applicable, are polynomials. Then* (1) *possesses at most* j *linearly independent solutions of finite order.*

Sharpness of Frei's theorem

Example. The functions

$$\begin{aligned} f_1(z) &= e^z + z \\ f_2(z) &= e^z - 1 \\ f_3(z) &= z + 1 \end{aligned}$$

are solutions of

$$f''' + \left(z - 1 + e^{-z}\right) f'' - (z+1)f' + f = 0,$$

and any two of them are linearly independent. This illustrates the sharpness of Frei's theorem in the case $n = 3$.

Frei's example in the second order case

The equation

$$f'' + e^{-z} f' + \alpha f = 0$$

where $\alpha \neq 0$ is a constant, has a subnormal solution if and only if $\alpha = -m^2$ for a positive integer m. The subnormal solution is a polynomial of degree m in e^z, that is

$$f(z) = C_0 + C_1 e^z + \cdots + C_m e^{mz},$$

where $C_0, \ldots, C_m \in \mathbb{C}$ with $C_m \neq 0$. In fact, $C_j \neq 0$ for $0 \leq j \leq m$ holds, but requires a short proof.

Examples of third order equations

- The function $f(z)=e^{-z}+z-1$ satisfies

$$f'''+(e^z-z)f''-zf'+f=0.$$

- The function $f(z)=e^z-1$ satisfies

$$f'''-2f''+e^{-z}f'+f=0.$$

- The function $f(z)=16-27e^{-2z}+27e^{-3z}$ satisfies

$$f'''+(1/9)(9+9e^z+4e^{2z})f''-5f'+3f=0.$$

Examples of solutions of order two

- The function $f(z) = \exp\left(z^2\right) - 1$ solves the following two equations:

$$\begin{aligned} f''' + \left(\exp\left(-z^2\right) - 2z - 1\right) f'' - 4f' + \left(4z^2 + 2\right) f &= 0, \\ f''' - 2zf'' - (2 + 2e^{-z^2})f' - 4zf &= 0. \end{aligned}$$

- The function $f(z) = \exp\left(z^2/2 + z\right) + z + 1$ is a solution of

$$f''' + \left(\exp\left(-z^2/2 - z\right) - z - 1\right) f'' - f' - (z+1)f = 0.$$

Examples of higher order equations

- If $P(z)$ and $Q(z)$ are any polynomials, then $f(z) = e^{-z} + 1$ solves

$$f^{(5)}+P(z)f^{(4)}+(1+P(z))f'''+Q(z)f''+(Q(z)+2e^{z})f'+2f = 0.$$

- Let n be an even number and μ be an integer such that $0 < \mu < n$. If $a_\mu(z) = e^{-z}$, $a_j = (-1)^j$ for $\mu < j < n$ and $a_j = (-1)^{j+1}$ for $0 \leq j < \mu$, then $f(z) = e^{z} + 1$ solves the equation

$$f^{(n)} + a_{n-1}f^{(n-1)} + \cdots + e^{-z}f^{(\mu)} + \cdots + a_1 f' + a_0 f = 0.$$

Standard form and normalized form

- An exp poly is an entire function of the form

$$f(z) = P_1(z)e^{Q_1(z)} + \cdots + P_k(z)e^{Q_k(z)},$$

where P_j's and Q_j's are polynomials in z.

- The constant $q = \max\{\deg(Q_j)\}$ is the order of f. If $q = 1$, then f is called an exponential sum.

- The normalized form of f is

$$f(z) = H_0(z) + H_1(z)e^{w_1 z^q} + \cdots + H_m(z)e^{w_m z^q},$$

where $H_j(z)$'s are either exp polys of order $< q$ or ordinary polys in z, the coefficients w_j are pairwise distinct, and $m \leq k$.

Dual exponential polynomials

- Suppose that f is an exp poly in the normalized form. If the nonzero conjugate leading coefficients $\overline{w}_1, \ldots, \overline{w}_m$ of f all lie on some ray $\arg(z) = \theta$, then f is called *a simple exp poly*.
- If g is another simple exponential polynomial such that $\rho(g) = \rho(f)$, where the non-zero conjugate leading coefficients of g all lie on the opposite ray $\arg(z) = \theta + \pi$, then f and g are called *dual exp polys*.
- For example, $f(z) = e^z + e^{2z} + e^{5z}$ and $g(z) = 1 + e^{-4z}$ are dual exp polys.

Second order case

Theorem

Suppose that f is an exp poly solution of

$$f'' + A(z)f' + B(z)f = 0,$$

where $A(z)$ and $B(z)$ are exp polys satisfying $\rho(B) < \rho(A)$. Then f and $A(z)$ are dual exp polys of order $q \in \mathbb{N}$.

In particular, if $\rho(Af') < q$, then $q = 1$ and

$$f(z) = c + \beta e^{\alpha z}, \quad A(z) = \gamma e^{-\alpha z} \quad \text{and} \quad B(z) = \mu,$$

where $\alpha, \beta, \gamma, \mu \in \mathbb{C} \setminus \{0\}$.

General order case

Theorem

Suppose that f is an exp poly solution of

$$f^{(n)} + a_{n-1}(z)f^{(n-1)} + \cdots + a_1(z)f' + a_0(z)f = 0,$$

where $a_j(z)$ are exp polys such that for precisely one index $\mu \in \{1, \cdots, n-1\}$, we have $\rho(a_j) < \rho(a_\mu)$ for all $j \neq \mu$. Then either f is a polynomial of degree $\leq \mu - 1$ or f and $a_\mu(z)$ are dual exp polys of order $q \in \mathbb{N}$.

In particular, if $\rho(a_\mu f^{(\mu)}) < q$ and $a_j(z)$ are polynomials for $j \neq \mu$, then

$$f(z) = S(z) + Q(z)e^{P(z)} \quad \text{and} \quad a_\mu(z) = R(z)e^{-P(z)},$$

where $P(z), Q(z), R(z), S(z)$ are polynomials and $\deg(P) = q$.

Finite order solutions

Theorem

Suppose that f is a **finite order solution** *of*

$$f^{(n)} + a_{n-1}(z)f^{(n-1)} + \cdots + a_1(z)f' + a_0(z)f = 0,$$

where $a_\mu(z)$ is a transcendental exp poly for precisely one index $\mu \in \{1, \cdots, n-1\}$, while $a_j(z)$ $(j \neq \mu)$ are poly's. Then either f is a poly of degree $\leq \mu - 1$ or $\rho(f) \geq \rho(a_\mu)$. In addition:

(a) *If $|a_\mu(z)|$ blows up exponentially in a sector S_1, then f has at most a polynomial growth in S_1.*

(b) *If $|a_\mu(z)|$ decays to zero in a sector S_2, then*

$$\log^+ |f(z)| = O\left(|z|^{1+\max_{j\neq\mu}\left\{\frac{\deg(a_j)}{n-j}\right\}}\right), \quad z \in S_2.$$

Tools for proofs

- General growth estimates for solutions
- Phragmén-Lindelöf principle
- Estimates for log derivatives and inverse log derivatives
- Steinmetz' result for quotients of exp polynomials
- Careful treatment of indicator diagrams
- Borel's lemma