Hyponormal Toeplitz Operators with Non-harmonic Symbols Acting on the Bergman Space

Matthew Fleeman and Constanze Liaw

CAFT 2018 - University of Crete

July 3, 2018

Hyponormal Operators

Let H be a complex Hilbert space and T be a bounded linear operator with adjoint T^*.

Hyponormal Operators

Let H be a complex Hilbert space and T be a bounded linear operator with adjoint T^*.

T is said to be *hyponormal* if $[T^*, T] := T^*T - TT^* \geq 0$. That is, if for all $u \in H$,

$$\langle [T^*, T]u, u \rangle \geq 0.$$

Hyponormal Operators

Let H be a complex Hilbert space and T be a bounded linear operator with adjoint T^*.

T is said to be *hyponormal* if $[T^*, T] := T^*T - TT^* \geq 0$. That is, if for all $u \in H$,

$$\langle [T^*, T]u, u \rangle \geq 0.$$

Used to study

- Spectral and perturbation theories of Hilbert space operators

Hyponormal Operators

Let H be a complex Hilbert space and T be a bounded linear operator with adjoint T^*.

T is said to be *hyponormal* if $[T^*, T] := T^*T - TT^* \geq 0$. That is, if for all $u \in H$,

$$\langle [T^*, T]u, u \rangle \geq 0.$$

Used to study

- Spectral and perturbation theories of Hilbert space operators
- Singular integral equations

Hyponormal Operators

Let H be a complex Hilbert space and T be a bounded linear operator with adjoint T^*.

T is said to be *hyponormal* if $[T^*, T] := T^*T - TT^* \geq 0$. That is, if for all $u \in H$,

$$\langle [T^*, T]u, u \rangle \geq 0.$$

Used to study

- Spectral and perturbation theories of Hilbert space operators
- Singular integral equations
- Scattering theory

Hyponormal Operators

Let H be a complex Hilbert space and T be a bounded linear operator with adjoint T^*.

T is said to be *hyponormal* if $[T^*, T] := T^*T - TT^* \geq 0$. That is, if for all $u \in H$,

$$\langle [T^*, T]u, u \rangle \geq 0.$$

Used to study

- Spectral and perturbation theories of Hilbert space operators
- Singular integral equations
- Scattering theory

Self-adjoint $\Longrightarrow$ Normal $\Longrightarrow$ Sub-normal $\Longrightarrow$ Hyponormal

Putnam's Inequality

One particularly interesting result for hyponormal operators is Putnam's inequality.

Putnam's Inequality

One particularly interesting result for hyponormal operators is Putnam's inequality.

Theorem (C.R. Putnam, 1970)

If T is hyponormal then

$$\|[T^*, T]\| \leq \frac{\mathrm{Area}(\sigma(T))}{\pi},$$

where $\sigma(T)$ denotes the spectrum of T.

Putnam's Inequality

One particularly interesting result for hyponormal operators is Putnam's inequality.

Theorem (C.R. Putnam, 1970)

If T is hyponormal then

$$\|[T^*, T]\| \leq \frac{\mathrm{Area}(\sigma(T))}{\pi},$$

where $\sigma(T)$ denotes the spectrum of T.

We are interested in studying the stability of hyponormal operators under perturbation in certain analytic function spaces.

The Hardy Space

Definition

A function $f(z)$, analytic in $\mathbb{D}$, is said to belong to the *Hardy space*, H^2, if

$$\sup_{0<r<1} \int_{\mathbb{T}} |f(re^{i\theta})|^2 \, d\theta < \infty$$

The Hardy Space

Definition

A function $f(z)$, analytic in $\mathbb{D}$, is said to belong to the *Hardy space*, H^2, if

$$\sup_{0<r<1} \int_{\mathbb{T}} |f(re^{i\theta})|^2 d\theta < \infty$$

H^2 can be thought of as a subspace of $L^2(\mathbb{T})$.

The Hardy Space

Definition

A function $f(z)$, analytic in $\mathbb{D}$, is said to belong to the *Hardy space*, H^2, if

$$\sup_{0<r<1} \int_{\mathbb{T}} |f(re^{i\theta})|^2 d\theta < \infty$$

H^2 can be thought of as a subspace of $L^2(\mathbb{T})$.

Definition

Let $\varphi(z)$ be in $L^\infty(\mathbb{T})$. The Toeplitz operator $T_\varphi : H^2 \to H^2$ with symbol φ is given by

$$T_\varphi f = P_+(\varphi f),$$

where P_+ is the projection from $L^2(\mathbb{T})$ onto H^2.

Theorem (C. Cowen, 1988)

Let $\varphi \in L^\infty(\mathbb{T})$ be given by $\varphi = f + \bar{g}$, with $f, g \in H^2$. Then T_φ is hyponormal if and only if

$$g = c + T_{\bar{h}} f,$$

for some constant c and some $h \in H^\infty(\mathbb{D})$, with $\|h\|_\infty \leq 1$.

Hyponormal Operators in the Hardy Space

Theorem (C. Cowen, 1988)

Let $\varphi \in L^\infty(\mathbb{T})$ be given by $\varphi = f + \bar{g}$, with $f, g \in H^2$. Then T_φ is hyponormal if and only if

$$g = c + T_{\bar{h}} f,$$

for some constant c and some $h \in H^\infty(\mathbb{D})$, with $\|h\|_\infty \leq 1$.

The proof relies on a dilation theorem by Sarason and the fact that $H^{2\perp}$ consists of conjugates of functions in zH^2.

The Bergman Space

Definition

A function f analytic in $\mathbb{D}$ is said to belong to the *Bergman space*, A^2, if

$$\frac{1}{\pi}\int_{\mathbb{D}} |f(z)|^2 \, dA(z) < \infty,$$

where dA is area measure on $\mathbb{D}$.

The Bergman Space

Definition

A function f analytic in $\mathbb{D}$ is said to belong to the *Bergman space*, A^2, if

$$\frac{1}{\pi}\int_{\mathbb{D}} |f(z)|^2 \, dA(z) < \infty,$$

where dA is area measure on $\mathbb{D}$.

For $f = \sum_{n\geq 0} a_n z^n$ in $A^2(\mathbb{D})$, we have that

$$\|f\|_{A^2}^2 = \sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1}.$$

Toeplitz Operators on A^2

Definition

Let $\varphi(z)$ be a bounded function in $\mathbb{D}$. The Toeplitz operator $T_\varphi : A^2 \to A^2$ with symbol φ is given by

$$T_\varphi f = P(\varphi f),$$

where P is the projection from $L^2(\mathbb{D})$ onto A^2.

Toeplitz Operators on A^2

Definition

Let $\varphi(z)$ be a bounded function in $\mathbb{D}$. The Toeplitz operator $T_\varphi : A^2 \to A^2$ with symbol φ is given by

$$T_\varphi f = P(\varphi f),$$

where P is the projection from $L^2(\mathbb{D})$ onto A^2.

It is still an open question to completely classify hyponormal Toeplitz operators acting on A^2.

Toeplitz Operators on A^2

Definition

Let $\varphi(z)$ be a bounded function in $\mathbb{D}$. The Toeplitz operator $T_\varphi : A^2 \to A^2$ with symbol φ is given by

$$T_\varphi f = P(\varphi f),$$

where P is the projection from $L^2(\mathbb{D})$ onto A^2.

It is still an open question to completely classify hyponormal Toeplitz operators acting on A^2.

There is no analog of Sarason's dilation theorem.

Toeplitz Operators on A^2

Definition

Let $\varphi(z)$ be a bounded function in $\mathbb{D}$. The Toeplitz operator $T_\varphi : A^2 \to A^2$ with symbol φ is given by

$$T_\varphi f = P(\varphi f),$$

where P is the projection from $L^2(\mathbb{D})$ onto A^2.

It is still an open question to completely classify hyponormal Toeplitz operators acting on A^2.

There is no analog of Sarason's dilation theorem.

$(A^2)^\perp$ is a much larger space.

Some useful facts about Toeplitz operators

- $T_\varphi^* = T_{\bar{\varphi}}$

Some useful facts about Toeplitz operators

- $T_{\varphi}^{*} = T_{\bar{\varphi}}$
- $T_{f+g} = T_f + T_g$

Some useful facts about Toeplitz operators

- $T_\varphi^* = T_{\bar{\varphi}}$
- $T_{f+g} = T_f + T_g$
- T_φ hyponormal $\iff \|T_\varphi u\|^2 - \|T_{\overline{\varphi}} u\|^2 \geq 0$ for all $u \in A^2$.

Some useful facts about Toeplitz operators

- $T_\varphi^* = T_{\bar{\varphi}}$
- $T_{f+g} = T_f + T_g$
- T_φ hyponormal $\iff \|T_\varphi u\|^2 - \|T_{\overline{\varphi}} u\|^2 \geq 0$ for all $u \in A^2$.
- For $f, g \in L^\infty(\mathbb{D})$, and $u \in A^2$, we have that

$$\langle [T_{f+g}^*, T_{f+g}]u, u\rangle = \left(\|T_f u\|^2 - \|T_f^* u\|^2\right) + \left(\|T_g u\|^2 - \|T_g^* u\|^2\right)$$

$$+2\mathrm{Re}\left(\langle T_f u, T_g u\rangle - \langle T_f^* u, T_g^* u\rangle\right).$$

Known Results

Theorem (H. Sadraoui, 1992)

Let f and g be bounded analytic functions, such that $f' \in H^2$. If $T_{f+\bar{g}}$ acting on A^2 is hyponormal, then $g' \in H^2$ and $|g'| \leq |f'|$ almost everywhere on $\mathbb{T}$.

Known Results

Theorem (H. Sadraoui, 1992)

Let f and g be bounded analytic functions, such that $f' \in H^2$. If $T_{f+\bar{g}}$ acting on A^2 is hyponormal, then $g' \in H^2$ and $|g'| \leq |f'|$ almost everywhere on $\mathbb{T}$.

Interestingly, this is a boundary value result!

Known Results

Theorem (H. Sadraoui, 1992)

Let f and g be bounded analytic functions, such that $f' \in H^2$. If $T_{f+\bar{g}}$ acting on A^2 is hyponormal, then $g' \in H^2$ and $|g'| \leq |f'|$ almost everywhere on $\mathbb{T}$.

Interestingly, this is a boundary value result!

P. Ahern and Z. Čučković showed in 1996 that the hypotheses can be relaxed quite a bit.

Known results continued

The condition is necessary, but not sufficient in general, as demonstrated by the next theorem.

Known results continued

The condition is necessary, but not sufficient in general, as demonstrated by the next theorem.

Theorem (H. Sadraoui, 1992)

1. *If* $m \leq n$*, then* $T_{z^n+\alpha\bar{z}^m}$ *is hyponormal if and only if* $|\alpha| \leq \sqrt{\frac{m+1}{n+1}}$.

2. *If* $m \geq n$*,* $T_{z^n+\alpha\bar{z}^m}$ *is hyponormal if and only if* $|\alpha| \leq \frac{n}{m}$.

Known results continued

The condition is necessary, but not sufficient in general, as demonstrated by the next theorem.

Theorem (H. Sadraoui, 1992)

1. *If $m \leq n$, then $T_{z^n+\alpha\bar{z}^m}$ is hyponormal if and only if* $|\alpha| \leq \sqrt{\frac{m+1}{n+1}}$.

2. *If $m \geq n$, $T_{z^n+\alpha\bar{z}^m}$ is hyponormal if and only if* $|\alpha| \leq \frac{n}{m}$.

This leads to a host of examples where $|g'| \leq |f'|$ on $\mathbb{T}$, but $T_{f+\bar{g}}$ is not hyponormal.

Known results continued

The condition is necessary, but not sufficient in general, as demonstrated by the next theorem.

Theorem (H. Sadraoui, 1992)

1. *If* $m \leq n$, *then* $T_{z^n+\alpha\bar{z}^m}$ *is hyponormal if and only if* $|\alpha| \leq \sqrt{\frac{m+1}{n+1}}$.

2. *If* $m \geq n$, $T_{z^n+\alpha\bar{z}^m}$ *is hyponormal if and only if* $|\alpha| \leq \frac{n}{m}$.

This leads to a host of examples where $|g'| \leq |f'|$ on $\mathbb{T}$, but $T_{f+\bar{g}}$ is not hyponormal.

e.g. $T_{z^3+\bar{z}^2}$ is not hyponormal.

Known results continued

Theorem (I.S. Hwang and J. Lee, 2005)

Let $f(z) = a_m z^m + a_n z^n$ *and* $g(z) = a_{-m} z^m + a_{-n} z^n$, *with* $0 < m < n$. *If* $T_{f+\bar{g}}$ *is hyponormal and* $|a_n| \leq |a_{-n}|$ *then we have that*

$$n^2 |a_{-n}|^2 + m^2 |a_{-m}|^2 \leq m^2 |a_m|^2 + n^2 |a_n|^2$$

Known results continued

Theorem (I.S. Hwang and J. Lee, 2005)

Let $f(z) = a_m z^m + a_n z^n$ *and* $g(z) = a_{-m} z^m + a_{-n} z^n$*, with* $0 < m < n$*. If* $T_{f+\bar{g}}$ *is hyponormal and* $|a_n| \leq |a_{-n}|$ *then we have that*

$$n^2 |a_{-n}|^2 + m^2 |a_{-m}|^2 \leq m^2 |a_m|^2 + n^2 |a_n|^2$$

Theorem (Z. Čučković and R. Curto, 2016)

Suppose T_φ *is hyponormal on* $A^2(\mathbb{D})$ *with* $\varphi(z) = \alpha z^m + \beta z^n + \gamma \bar{z}^p + \delta \bar{z}^q$*, where* $m < n$ *and* $p < q$*, and* $\alpha, \beta, \gamma.\delta \in \mathbb{C}$*. Assume also that* $n - m = q - p$*. Then*

$$|\alpha|^2 n^2 + |\beta|^2 m^2 - |\gamma|^2 p^2 - |\delta|^2 q^2 \geq 2 |\bar{\alpha}\beta mn - \bar{\gamma}\delta pq| .$$

Known results continued

Theorem (I.S. Hwang and J. Lee, 2005)

Let $f(z) = a_m z^m + a_n z^n$ *and* $g(z) = a_{-m} z^m + a_{-n} z^n$, *with* $0 < m < n$. *If* $T_{f+\bar{g}}$ *is hyponormal and* $|a_n| \leq |a_{-n}|$ *then we have that*

$$n^2 |a_{-n}|^2 + m^2 |a_{-m}|^2 \leq m^2 |a_m|^2 + n^2 |a_n|^2$$

Theorem (Z. Čučković and R. Curto, 2016)

Suppose T_φ *is hyponormal on* $A^2(\mathbb{D})$ *with* $\varphi(z) = \alpha z^m + \beta z^n + \gamma \bar{z}^p + \delta \bar{z}^q$, *where* $m < n$ *and* $p < q$, *and* $\alpha, \beta, \gamma.\delta \in \mathbb{C}$. *Assume also that* $n - m = q - p$. *Then*

$$|\alpha|^2 n^2 + |\beta|^2 m^2 - |\gamma|^2 p^2 - |\delta|^2 q^2 \geq 2 |\bar{\alpha}\beta mn - \bar{\gamma}\delta pq| .$$

Note that so far, all the symbols involved are harmonic.

Small excursions into non-harmonic symbols

It is relatively straightforward to show that $T_{z^m \bar{z}^n}$ is hyponormal if and only if $m \geq n$.

Small excursions into non-harmonic symbols

It is relatively straightforward to show that $T_{z^m \bar{z}^n}$ is hyponormal if and only if $m \geq n$.

Even when the symbol is very "nice", hyponormality is not guaranteed.

Small excursions into non-harmonic symbols

It is relatively straightforward to show that $T_{z^m \bar{z}^n}$ is hyponormal if and only if $m \geq n$.

Even when the symbol is very "nice", hyponormality is not guaranteed.

Example

$T_{z - 2\sqrt{2}|z|^2}$ is not hyponormal. In particular,

$$\left\langle \left[T^*_{z-2\sqrt{2}|z|^2}, T_{z-2\sqrt{2}|z|^2} \right] \left(\frac{1}{2} + \frac{z}{\sqrt{2}} \right), \frac{1}{2} + \frac{z}{\sqrt{2}} \right\rangle < 0.$$

.

Small excursions into non-harmonic symbols

It is relatively straightforward to show that $T_{z^m\bar{z}^n}$ is hyponormal if and only if $m \geq n$.

Even when the symbol is very "nice", hyponormality is not guaranteed.

Example

$T_{z-2\sqrt{2}|z|^2}$ is not hyponormal. In particular,

$$\left\langle \left[T^*_{z-2\sqrt{2}|z|^2}, T_{z-2\sqrt{2}|z|^2}\right]\left(\frac{1}{2}+\frac{z}{\sqrt{2}}\right), \frac{1}{2}+\frac{z}{\sqrt{2}}\right\rangle < 0.$$

.

In fact $T_{\frac{z}{C}+|z|^2}$ fails to be hyponormal whenever $|C| \geq 2\sqrt{2}$!

Two term non-harmonic polynomial symbols

We look at when two-term non-harmonic polynomials can be the symbol of a hyponormal operator.

Two term non-harmonic polynomial symbols

We look at when two-term non-harmonic polynomials can be the symbol of a hyponormal operator.

Theorem (MCF and Liaw, 2017)

Suppose $\varphi = \alpha z^m \bar{z}^n + z^i \bar{z}^j$, with $m > n$ and $m - n > i - j$. Then T_φ is hyponormal if α lies outside some annulus (when $i > j$) or outside some disk (when $j > i$), which depends on m, n, i, and j.

Two term non-harmonic polynomial symbols

We look at when two-term non-harmonic polynomials can be the symbol of a hyponormal operator.

Theorem (MCF and Liaw, 2017)

Suppose $\varphi = \alpha z^m \bar{z}^n + z^i \bar{z}^j$, with $m > n$ and $m - n > i - j$. Then T_φ is hyponormal if α lies outside some annulus (when $i > j$) or outside some disk (when $j > i$), which depends on m, n, i, and j.

The case when $m - n = i - j$ is not covered by this theorem, but will be addressed later.

Two term non-harmonic polynomial symbols

We can use this to construct hyponormal operators.

Two term non-harmonic polynomial symbols

We can use this to construct hyponormal operators.

Example (MCF and Liaw, 2017)

Consider $\varphi(z) = z^2\overline{z} + \frac{1}{7}\overline{z}^4 z^3$. By checking against the conditions from the previous Theorem, we can show that T_φ is hyponormal

Two term non-harmonic polynomial symbols

We can use this to construct hyponormal operators.

Example (MCF and Liaw, 2017)

Consider $\varphi(z) = z^2\bar{z} + \frac{1}{7}\bar{z}^4 z^3$. By checking against the conditions from the previous Theorem, we can show that T_φ is hyponormal

This example can be generalized.

Two term non-harmonic polynomial symbols

We can use this to construct hyponormal operators.

Example (MCF and Liaw, 2017)

Consider $\varphi(z) = z^2\bar{z} + \frac{1}{7}\bar{z}^4 z^3$. By checking against the conditions from the previous Theorem, we can show that T_φ is hyponormal

This example can be generalized.

Theorem (MCF and Liaw, 2017)

Fix $\delta \in \mathbb{N}$. For every integer $n \in \mathbb{N}$ there exists $j \in \mathbb{N}$, such that T_φ with symbol $\varphi(z) = z^{n+\delta}\bar{z}^n + \frac{1}{2j+\delta}\bar{z}^{j+\delta} z^j$ is hyponormal.

Two term non-harmonic polynomial symbols

We can use this to construct hyponormal operators.

Example (MCF and Liaw, 2017)

Consider $\varphi(z) = z^2\overline{z} + \frac{1}{7}\overline{z}^4 z^3$. By checking against the conditions from the previous Theorem, we can show that T_φ is hyponormal

This example can be generalized.

Theorem (MCF and Liaw, 2017)

Fix $\delta \in \mathbb{N}$. For every integer $n \in \mathbb{N}$ there exists $j \in \mathbb{N}$, such that T_φ with symbol $\varphi(z) = z^{n+\delta}\overline{z}^n + \frac{1}{2j+\delta}\overline{z}^{j+\delta} z^j$ is hyponormal.

Up until now everything has been in terms of the moduli of the coefficients.

Mellin Transform

Definition

Suppose $\varphi \in L^1([0,1], rdr)$. For $\mathrm{Re}\, z \geq 2$,the *Mellin Transform* of φ, is given by

$$\hat{\varphi}(z) = \int_0^1 \varphi(x) x^{z-1} dx$$

Mellin Transform

Definition

Suppose $\varphi \in L^1\left([0,1], rdr\right)$. For $\operatorname{Re} z \geq 2$,the *Mellin Transform* of φ, is given by

$$\hat{\varphi}(z) = \int_0^1 \varphi(x) x^{z-1} dx$$

For $\varphi(re^{i\theta}) = e^{ik\theta}\varphi_0(r)$, with $k \in \mathbb{Z}$ and φ_0 radial,

$$T_\varphi z^n = \begin{cases} 2\left(n+k+1\right)\hat{\varphi}_0(2n+k+2) z^{n+k} & n+k \geq 0 \\ 0 & n+k < 0 \end{cases}$$

and

$$T_{\bar{\varphi}} z^n = \begin{cases} 2\left(n-k+1\right)\hat{\varphi}_0(2n-k+2) z^{n-k} & n-k \geq 0 \\ 0 & n-k < 0 \end{cases}$$

A more general result

Theorem (Y. Lu and C. Liu, 2009)

Let $\varphi(re^{i\theta}) = e^{i\delta\theta}\varphi_0(r) \in L^\infty(\mathbb{D})$, where $\delta \in \mathbb{Z}$ and φ_0 is radial. Then T_φ is hyponormal if and only if one of the following conditions holds:

A more general result

Theorem (Y. Lu and C. Liu, 2009)

Let $\varphi(re^{i\theta}) = e^{i\delta\theta}\varphi_0(r) \in L^\infty(\mathbb{D})$, *where* $\delta \in \mathbb{Z}$ *and* φ_0 *is radial. Then* T_φ *is hyponormal if and only if one of the following conditions holds:*

1) $\delta < 0$ *and* $\varphi_0 \equiv 0$;

A more general result

Theorem (Y. Lu and C. Liu, 2009)

Let $\varphi(re^{i\theta}) = e^{i\delta\theta}\varphi_0(r) \in L^\infty(\mathbb{D})$, *where* $\delta \in \mathbb{Z}$ *and* φ_0 *is radial. Then* T_φ *is hyponormal if and only if one of the following conditions holds:*

1) $\delta < 0$ *and* $\varphi_0 \equiv 0$;
2) $\delta = 0$;

A more general result

Theorem (Y. Lu and C. Liu, 2009)

Let $\varphi(re^{i\theta}) = e^{i\delta\theta}\varphi_0(r) \in L^\infty(\mathbb{D})$, *where* $\delta \in \mathbb{Z}$ *and* φ_0 *is radial. Then* T_φ *is hyponormal if and only if one of the following conditions holds:*

1) $\delta < 0$ *and* $\varphi_0 \equiv 0$;
2) $\delta = 0$;
3) $\delta > 0$ *and for each* $\alpha \geq \delta$,

$$|\widehat{\varphi}_0(2\alpha + \delta + 2)| \geq \sqrt{\frac{\alpha - \delta + 1}{\alpha + \delta + 1}}\,|\widehat{\varphi}_0(2\alpha - \delta + 2)|\,. \qquad (1)$$

A consequence of the Liu-Lu Theorem

From this Theorem, we may conclude that if

$$\varphi(z) = a_1 z^{m_1} \bar{z}^{n_1} + \ldots + a_k z^{m_k} \bar{z}^{n_k},$$

with $m_1 - n_1 = \ldots = m_k - n_k \geq 0$, and a_i all lie on the same ray for $1 \leq i \leq k$, then T_φ is hyponormal.

A consequence of the Liu-Lu Theorem

From this Theorem, we may conclude that if

$$\varphi(z) = a_1 z^{m_1} \bar{z}^{n_1} + \ldots + a_k z^{m_k} \bar{z}^{n_k},$$

with $m_1 - n_1 = \ldots = m_k - n_k \geq 0$, and a_i all lie on the same ray for $1 \leq i \leq k$, then T_φ is hyponormal.

- If we take $\delta = m_1 - n_1$, we may write

$$\varphi(re^{i\theta}) = e^{i\delta\theta} \left(a_1 r^{m_1+n_1} + \ldots + a_k r^{m_k+n_k} \right).$$

A consequence of the Liu-Lu Theorem

From this Theorem, we may conclude that if

$$\varphi(z) = a_1 z^{m_1} \bar{z}^{n_1} + \ldots + a_k z^{m_k} \bar{z}^{n_k},$$

with $m_1 - n_1 = \ldots = m_k - n_k \geq 0$, and a_i all lie on the same ray for $1 \leq i \leq k$, then T_φ is hyponormal.

- If we take $\delta = m_1 - n_1$, we may write

$$\varphi(re^{i\theta}) = e^{i\delta\theta} \left(a_1 r^{m_1+n_1} + \ldots + a_k r^{m_k+n_k} \right).$$

- Since $T_{a_i z^{m_i} \bar{z}^{n_i}}$ is hyponormal for $1 \leq i \leq n$, then inequality (1) is satisfied for each i individually

A consequence of the Liu-Lu Theorem

From this Theorem, we may conclude that if

$$\varphi(z) = a_1 z^{m_1} \bar{z}^{n_1} + \ldots + a_k z^{m_k} \bar{z}^{n_k},$$

with $m_1 - n_1 = \ldots = m_k - n_k \geq 0$, and a_i all lie on the same ray for $1 \leq i \leq k$, then T_φ is hyponormal.

- If we take $\delta = m_1 - n_1$, we may write

$$\varphi(re^{i\theta}) = e^{i\delta\theta} \left(a_1 r^{m_1+n_1} + \ldots + a_k r^{m_k+n_k} \right).$$

- Since $T_{a_i z^{m_i} \bar{z}^{n_i}}$ is hyponormal for $1 \leq i \leq n$, then inequality (1) is satisfied for each i individually
- Since all a_i lie on the same ray inequality (1) will be satisfied by the sum.

Argument Matters

The requirement that all a_i lie on the same ray is not very satisfactory.

Argument Matters

The requirement that all a_i lie on the same ray is not very satisfactory.

We would like to relax this condition, but we cannot drop it entirely.

Argument Matters

The requirement that all a_i lie on the same ray is not very satisfactory.

We would like to relax this condition, but we cannot drop it entirely.

Example

Let $\varphi(z) = z^2\bar{z} - z^3\bar{z}^2$. Then $\widehat{\varphi}_0(k) = \frac{1}{k+3} - \frac{1}{k+5}$, and we find that

$$\frac{1}{2\alpha+6} - \frac{1}{2\alpha+8} < \sqrt{\frac{\alpha}{\alpha+2}}\left(\frac{1}{2\alpha+4} - \frac{1}{2\alpha+6}\right),$$

whenever $\alpha \geq 2$. By the Liu-Lu Theorem, T_φ cannot be hyponormal.

Argument Matters

The requirement that all a_i lie on the same ray is not very satisfactory.

We would like to relax this condition, but we cannot drop it entirely.

Example

Let $\varphi(z) = z^2\bar{z} - z^3\bar{z}^2$. Then $\widehat{\varphi}_0(k) = \frac{1}{k+3} - \frac{1}{k+5}$, and we find that

$$\frac{1}{2\alpha+6} - \frac{1}{2\alpha+8} < \sqrt{\frac{\alpha}{\alpha+2}}\left(\frac{1}{2\alpha+4} - \frac{1}{2\alpha+6}\right),$$

whenever $\alpha \geq 2$. By the Liu-Lu Theorem, T_φ cannot be hyponormal.

However if $\varphi(z) = z^2\bar{z} + z^3\bar{z}^2$, then T_φ is hyponormal.

Argument Matters

Theorem (MCF and Liaw, 2017)

Let $\varphi(z) = a_1 z^{m_1}\bar{z}^{n_1} + \ldots + a_k z^{m_k}\bar{z}^{n_k}$*, with* $m_1 - n_1 = \ldots = m_k - n_k = \delta \geq 0$*, and* a_i *all lying in the same quarter-plane* $1 \leq i \leq k$ *(i.e.* $\max_{1\leq i,j\leq k} |\arg(a_i) - \arg(a_j)| \leq \frac{\pi}{2}$*), then* T_φ *is hyponormal.*

Argument Matters

Theorem (MCF and Liaw, 2017)

Let $\varphi(z) = a_1 z^{m_1} \bar{z}^{n_1} + \ldots + a_k z^{m_k} \bar{z}^{n_k}$, with $m_1 - n_1 = \ldots = m_k - n_k = \delta \geq 0$, and a_i all lying in the same quarter-plane $1 \leq i \leq k$ (i.e. $\max_{1 \leq i,j \leq k} |\arg(a_i) - \arg(a_j)| \leq \frac{\pi}{2}$), then T_φ is hyponormal.

The proof involves examining the Mellin transform of φ, and then applying the Liu-Lu theorem.

Argument Matters

In the case of a two-term polynomial we can also get a partial converse.

Argument Matters

In the case of a two-term polynomial we can also get a partial converse.

Theorem (MCF and Liaw, 2017)

Let $\varphi(z) = a_1 z^m \bar{z}^n + a_2 z^i \bar{z}^j$, *with* $m - n = i - j = \delta \geq 0$. *If*

$$0 \leq \frac{|a_1|}{\alpha + m + 1} - \frac{|a_2|}{\alpha + i + 1} < \frac{\alpha - \delta + 1}{\alpha + \delta + 1} \left(\frac{|a_1|}{\alpha + n + 1} - \frac{|a_2|}{\alpha + j + 1} \right)$$

for some $\alpha \geq \delta$, *then* T_φ *is hyponormal if and only if* $|\arg(a_1) - \arg(a_2)| \leq \frac{\pi}{2}$.

Idea of the proof

- WLOG assume that $a_1 > 0$, and let $\theta = \arg(a_2)$.

Idea of the proof

- WLOG assume that $a_1 > 0$, and let $\theta = \arg(a_2)$.
- Recall that by the Liu-Lu Theorem, T_φ is hyponormal if and only if for each $\alpha \geq \delta$,

$$|\widehat{\varphi}_0(2\alpha + \delta + 2)| \geq \sqrt{\frac{\alpha - \delta + 1}{\alpha + \delta + 1}}\, |\widehat{\varphi}_0(2\alpha - \delta + 2)|\,.$$

Idea of the proof

- WLOG assume that $a_1 > 0$, and let $\theta = \arg(a_2)$.
- Recall that by the Liu-Lu Theorem, T_φ is hyponormal if and only if for each $\alpha \geq \delta$,

$$\left|\widehat{\varphi}_0(2\alpha+\delta+2)\right| \geq \sqrt{\frac{\alpha-\delta+1}{\alpha+\delta+1}}\left|\widehat{\varphi}_0(2\alpha-\delta+2)\right|.$$

- This is equivalent to the condition that for $\alpha \geq \delta$

$$F_{\varphi,\alpha}(\theta) := \left(\frac{a_1}{\alpha+m+1} + \frac{|a_2|\cos(\theta)}{\alpha+i+1}\right)^2 + \frac{|a_2|^2\sin^2(\theta)}{(\alpha+i+1)^2}$$
$$-\frac{\alpha-\delta+1}{\alpha+\delta+1}\left[\left(\frac{a_1}{\alpha+n+1} + \frac{|a_2|\cos(\theta)}{\alpha+j+1}\right)^2 + \frac{|a_2|^2\sin^2(\theta)}{(\alpha+j+1)^2}\right] \geq 0.$$

Idea of the proof

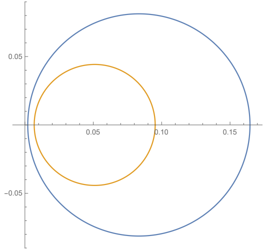

Figure: The situation when $\alpha = 6$, $m = 5$, $i = 9$, and $\delta = 4$

Consider the two circles:

$$C_1 := \left\{ z : \left| z - \frac{a_1}{\alpha+m+1} \right| = \frac{|a_2|}{\alpha+i+1} \right\}$$

$$C_2 := \left\{ z : \left| z - \frac{\alpha-\delta+1}{\alpha+\delta+1} \frac{a_1}{\alpha+n+1} \right| = \frac{\alpha-\delta+1}{\alpha+\delta+1} \frac{|a_2|}{\alpha+j+1} \right\}$$

Idea of the proof

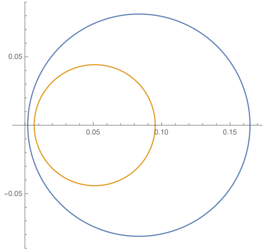

Figure: The situation when $\alpha = 6$, $m = 5$, $i = 9$, and $\delta = 4$

The hypothesis that

$$0 \leq \frac{|a_1|}{\alpha + m + 1} - \frac{|a_2|}{\alpha + i + 1} < \frac{\alpha - \delta + 1}{\alpha + \delta + 1}\left(\frac{|a_1|}{\alpha + n + 1} - \frac{|a_2|}{\alpha + j + 1}\right)$$

guarantees that C_2 lies completely in the region bounded by C_1.

Idea of the proof

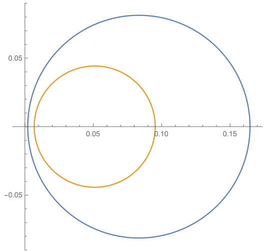

Figure: The situation when $\alpha = 6$, $m = 5$, $i = 9$, and $\delta = 4$

For every α there will exist a $\frac{\pi}{2} \leq \theta_\alpha < \pi$ such that $F_{\varphi,\alpha}(\theta) < 0$ for $\theta_\alpha < \theta < \pi$.

Idea of the proof

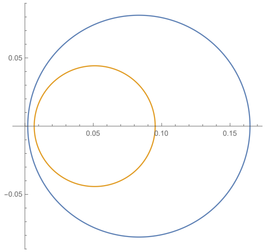

Figure: The situation when $\alpha = 6$, $m = 5$, $i = 9$, and $\delta = 4$

For every α there will exist a $\frac{\pi}{2} \leq \theta_\alpha < \pi$ such that $F_{\varphi,\alpha}(\theta) < 0$ for $\theta_\alpha < \theta < \pi$.

As $\alpha \to \infty$, we find that $\theta_\alpha \to \frac{\pi}{2}$, and so T_φ is hyponormal if and only if $|\theta| \leq \frac{\pi}{2}$.

Argument only matters sometimes

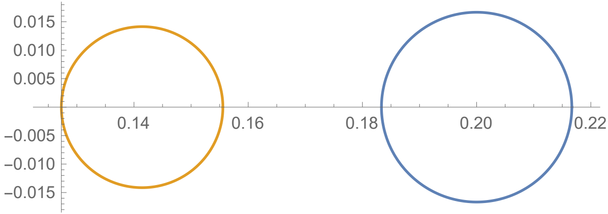

Figure: The situation when $\alpha = 2$, $m = 2$, $i = 3$, and $\delta = 1$

Let $\varphi_\theta(z) = \varphi(z) = z^2\bar{z} + \frac{1}{10}e^{i\theta}z^3\bar{z}^2$. As $\alpha \to \infty$, we find that $F_{\varphi,\alpha}(\theta) > 0$ for all $\theta \in [0, \pi]$ and all $\alpha \geq 1$.

Remarks and Further Research

We would like to thank Carl Cowen for his helpful correspondence, and Brian Simanek for very fruitful discussions.

Remarks and Further Research

We would like to thank Carl Cowen for his helpful correspondence, and Brian Simanek for very fruitful discussions.

The current proofs rely on rather straightforward calculations and "hard" analysis. We would like to find "softer", more function theoretic proofs, if possible, of these results including the Liu-Lu Theorem. Our current estimates could also be sharpened quite a bit.

Remarks and Further Research

We would like to thank Carl Cowen for his helpful correspondence, and Brian Simanek for very fruitful discussions.

The current proofs rely on rather straightforward calculations and "hard" analysis. We would like to find "softer", more function theoretic proofs, if possible, of these results including the Liu-Lu Theorem. Our current estimates could also be sharpened quite a bit.

We would also like to explore more qualitative conditions, similar to Sardraoui's results, on a symbol φ for when T_φ is hyponormal.

Remarks and Further Research

We would like to thank Carl Cowen for his helpful correspondence, and Brian Simanek for very fruitful discussions.

The current proofs rely on rather straightforward calculations and "hard" analysis. We would like to find "softer", more function theoretic proofs, if possible, of these results including the Liu-Lu Theorem. Our current estimates could also be sharpened quite a bit.

We would also like to explore more qualitative conditions, similar to Sardraoui's results, on a symbol φ for when T_φ is hyponormal.

For example, if $f, g \in C^\infty(\bar{\mathbb{D}})$ and T_{f+g} is hyponormal, does that imply a necessary relationship between $|f_z|$ and $|g_{\bar{z}}|$?

E$\upsilon\chi\alpha\rho\iota\sigma\tau\omega$!