Periodic cycles and singular values of entire transcendental functions

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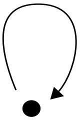
Holomorphic Dynamics

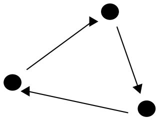
- We are interested in the dynamics generated by iteration of analytic maps on the complex plane.
  Examples:
  - Root finding algorithms (Newton's method, etc)
  - Complexification of real models, ...

- Main goal: To classify initial conditions in terms of the asymptotic behavior of their orbits

$$z_0, f(z_0), f^2(z_0), \cdots, f^n(z_0), \cdots$$

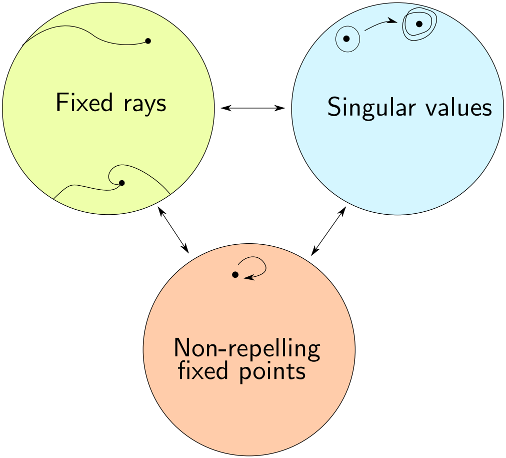
- Fixed (or periodic) points (equilibria of the system) are of special importance.





Plan

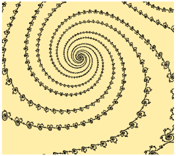
Given $f : \mathbb{C} \to \mathbb{C}$ holomorphic (i.e. f entire), we will find connections between three objects.



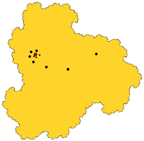
The discussion can be generalized to periodic rays and periodic points.

1. Fixed points

The multiplier of a fixed point z_0, $\rho = f'(z_0)$ (or $\rho = (f^p)'(z_0)$ if z_0 is p-periodic) gives information about its stability (the behaviour of nearby orbits).

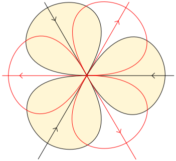


Repelling ($|\rho| > 1$)

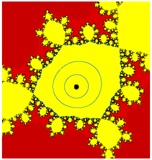


Attracting ($|\rho| < 1$)

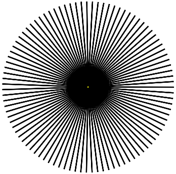
Indifferent if $\rho = e^{2\pi i\theta}$.



Parabolic ($\theta = p/q$)



Siegel (z_0 is stable)



Cremer (otherwise)

1. Fixed points

- Classical problem: Bounding and locating the number of non-repelling periodic points for a given dynamical system.

- Cremer points are the least understood of all types of fixed points. They introduce "bad" topological properties wherever they are.

Question 1

Can Cremer points lie on the boundary of an attracting basin (or parabolic basins, or Siegel disks)??

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2. Singular values

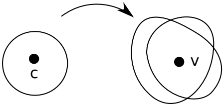
Holomorphic maps are local homeomorphisms everywhere except at the critical points

$$Crit(f) = \{c \mid f'(c) = 0\}.$$

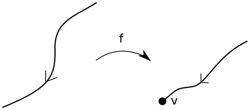
The set of singular values $S(f) = \mathrm{Sing}(f^{-1})$, consists of points for which some local branch of f^{-1} fails to be well defined.

These can be

- Critical values $CV = \{v = f(c) | c \in Crit(f)\}$;
- Asymptotic values $AV = \{a = \lim_{t \to \infty} f(\gamma(t)); \gamma(t) \to \infty\}$.



critical value



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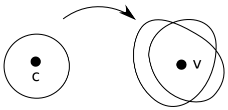

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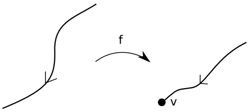

asymptotic value

2. Singular values

Why are they relevant?

Singular values play an important role because:

- Basins of attraction (of attracting or parabolic cycles) must contain at least one singular value.
- Cremer points and the boundary of Siegel disks must be accumulated by the orbit of at least one singular value.

BUT a priori, one singular orbit might accumulate in more than one non-repelling cycle!

Standing assumptions:

- $f : \mathbb{C} \to \mathbb{C}$ holomorphic (polynomial or transcendental)
- f of finite order and postsingularly bounded (PSB) (orbits of S are bounded).

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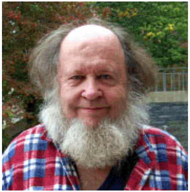
3. Fixed rays

- Rays are unbounded curves in the escaping set

$$I(f) = \{z \in \mathbb{C} \mid f^n(z) \to \infty\}.$$

- They provide a useful structure in the dynamical plane.
- They appear in a natural way when f is a polynomial. They also exist for entire transcendental functions if $f \in PSB$.



Adrien Douady

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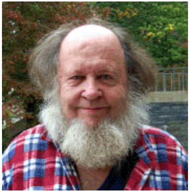

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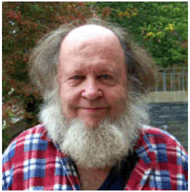

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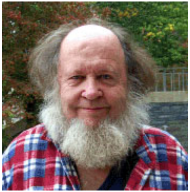

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3. Fixed rays (for polynomials)

If f is a PSB polynomial,

- ∞ is a superattracting fixed point, and $I(f)$ is its basin of attraction.
- $I(f)$ is open, connected and simply connected.
- f is conformally conjugate to z^d on $I(f)$

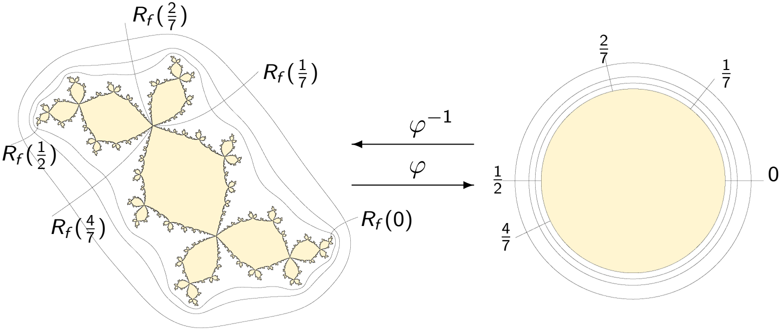
$$\begin{array}{ccc} I(f) & \xrightarrow{f} & I(f) \\ \varphi(conf)\downarrow & & \downarrow\varphi \\ \mathbb{C}\setminus\overline{\mathbb{D}} & \xrightarrow{z\mapsto z^d} & \mathbb{C}\setminus\overline{\mathbb{D}} \end{array}$$

- Hence $I(f)$ is folliated by rays

$$R_f(\theta) = \{\varphi^{-1}(\{\arg(z) = \theta\}); \theta \in \mathbb{R}/\mathbb{Z}\},$$

which obey the dynamics of multiplication by d (on angles),

3. Fixed rays (for polynomials)



$$f(R_f(\theta)) = R_f(d \cdot \theta).$$

All rational rays land, i.e. they have a limit point which is not escaping.

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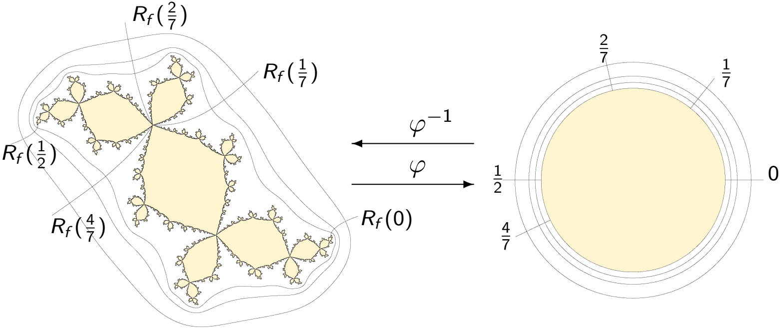

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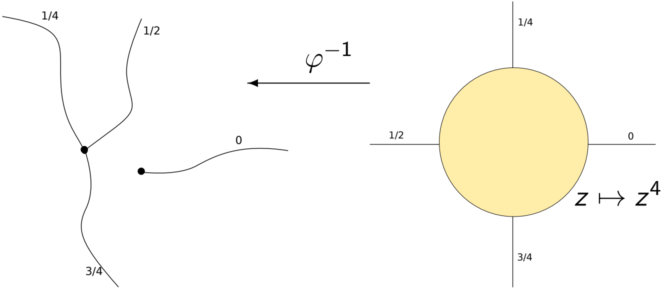
All rational rays land, i.e. they have a limit point which is not escaping.

3. Fixed rays (for polynomials)

- We will be interested in the $d-1$ fixed rays of f, i.e.

$$R_f(\theta) \quad \text{with } \theta \in \{0, \frac{1}{d-1}, \frac{2}{d-1}, \cdots, \frac{d-2}{d-1}\},$$

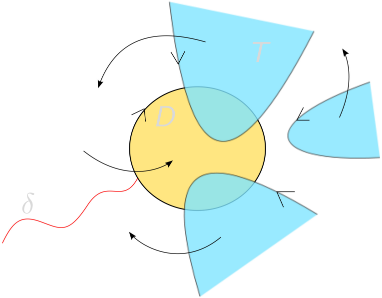
which must land at repelling or parabolic fixed points (Snail lemma).



3. Fixed rays (for entire transcendental functions)

$f \in PSB$. Let D be a closed *disk* containing $\mathrm{Sing}(f^{-1})$.

Connected components of $\mathcal{T} = f^{-1}(\mathbb{C} \setminus D)$ are called tracts, and are unbounded Jordan domains.



For all $T \subset \mathcal{T}$,
$f : T \to \mathbb{C} \setminus D$ is a universal covering.

Tracts cannot accumulate. $\Rightarrow$ finitely many tracts cut the disk D.

$\Rightarrow \exists$ a curve $\delta \subset \mathbb{C} \setminus D$ connecting ∂D with ∞.

R. L. Devaney and F. Tangerman. *Dynamics of entire functions near the essential singularity*. Ergodic Theory Dynam. Systems 6 (1986), 489-503..

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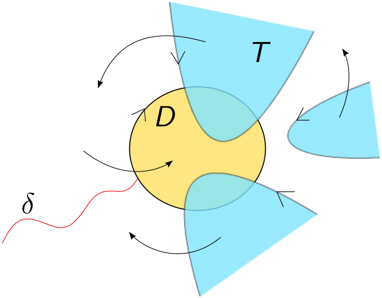

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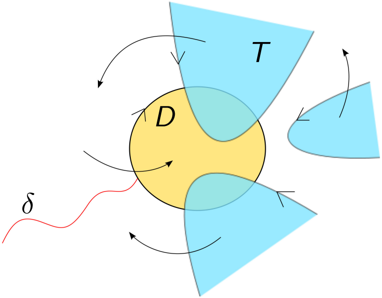

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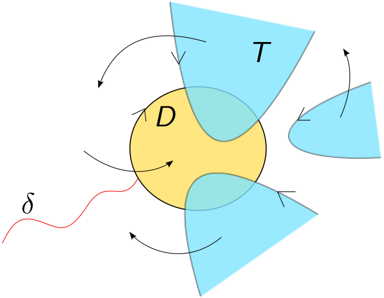

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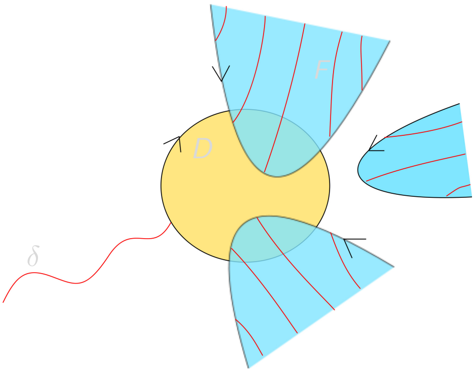
Inside each $T \subset \mathcal{T}$ consider the infinite collection of curves in $f^{-1}(\delta)$.



They divide T into fundamental domains. Let $\mathcal{F}$ be the union of all fundamental domains.

For each $F \subset \mathcal{F}$, $f : F \to \mathbb{C} \setminus (D \cup \delta)$ is conformal.

Observe this implies a behavior like $z \mapsto z^d$ when we cut d fundamental domains high enough. ☺

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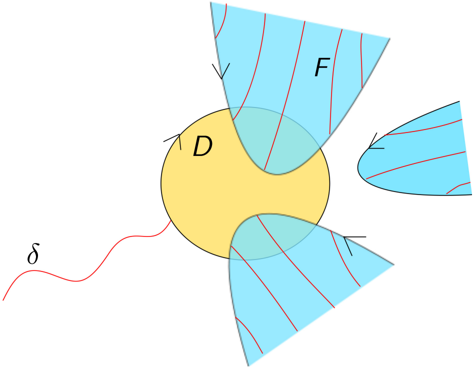

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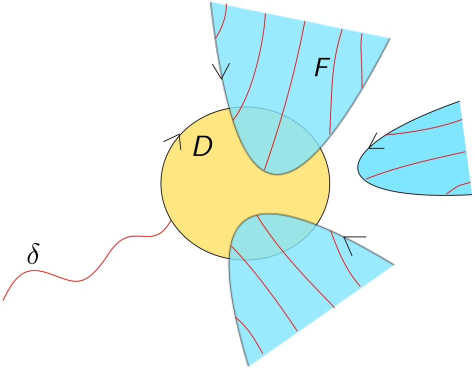

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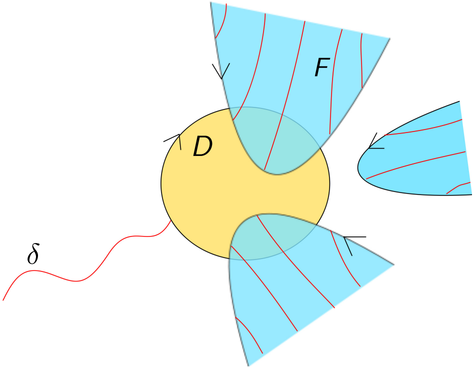

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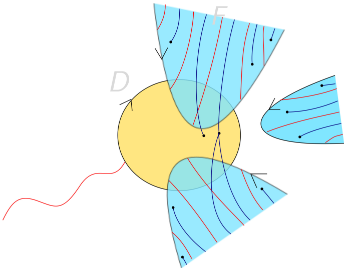
3. Fixed rays (for entire transcendental functions)

∃! fixed ray asymptotically contained in any fundamental domain.

That is, for each $F \subset \mathcal{F}$, ∃! a continuous invariant curve $R = R_F : (0, \infty) \to \mathbb{C}$ such that

1. $R(t) \to \infty$ as $t \to \infty$.
2. $f^n(R(t)) \to \infty$ as $n \to \infty$ for all $t > 0$.
3. $R(t) \in F$ for all $t > t_0$.

($f \in PSB$) All fixed rays land, i.e. $R : [0, \infty) \to \mathbb{C}$, and $R(0)$ is a repelling or parabolic fixed point.



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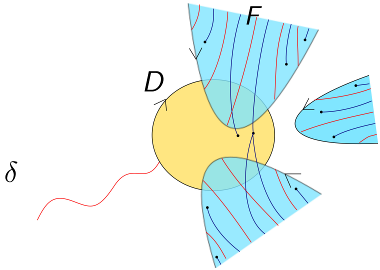

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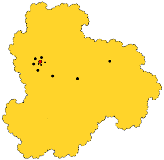
3. Fixed rays (for entire transcendental functions)

Example: $f(z) = \lambda e^z$
(for an appropriate value of λ).

3. Fixed rays (landing)

- Several fixed rays may share a landing fixed point.
- Otherwise we say that a ray lands alone.
- A fixed point might not be the landing point of any fixed ray. These are called interior fixed points and include all the non-repelling fixed points (except parabolics).



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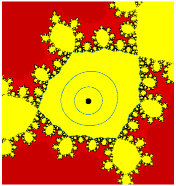

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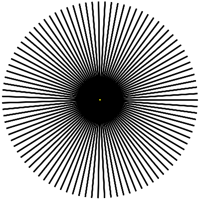

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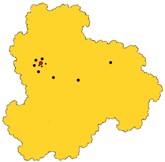

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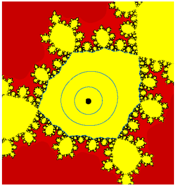

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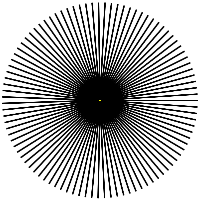

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Singular values and non-repelling cycles

Fatou-Shishikura inequality

Let $f : \mathbb{C} \to \mathbb{C}$ *be a polynomial. Then*

$$\#\{\textit{non-repelling cycles}\} \leq \# CP(f).$$

- Conjectured by Fatou in 1920.
- Uses quasiconformal surgery.
- Stronger version for rational maps.
- Alternative proofs by Douady and Hubbard'89 or Epstein'99 using quadratic differentials.
- Extension to entire transcendental maps (finite type) by Eremenko and Lyubich'92 and Goldberg-Keen'86 (c.f. Epstein'99).

M. Shishikura. *On the quasiconformal surgery of rational functions*. Ann. Sci. Ec. Norm. Sup. **20** (1987), 1-29.

Singular values and non-repelling cycles

The Fatou-Shishikura inequality

But one question still remained.

Question 2

Is there a singular value accumulating to each non-repelling cycle, and to no other cycle?

Theorem (Blokh et al'16)

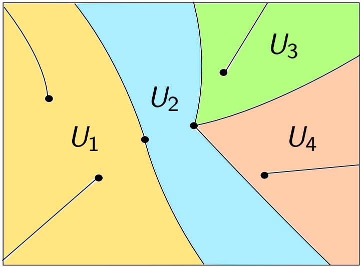
Let f be a polynomial. Then any non-repelling cycle is associated to a (weakly recurrent) critical point, and distinct non-repelling cycles are associated to distinct (weakly recurrent) critical points.

A. Blokh, D. Childers, G. Levin, L. Oversteegen, D. Schleicher. *An extended Fatou- Shishikura inequality and wandering branch continua for polynomials*. Adv. Math. **288** (2016), 1121-1174.

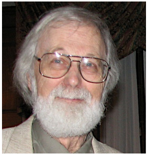
Fixed rays and interior fixed points

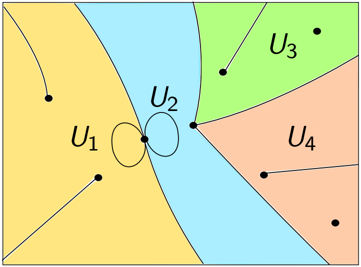
Goldberg–Milnor Separation Theorem ('93)

- f a PSB polynomial of degree d (i.e. with connected Julia set).
- $R_1, \ldots R_{d-1}$ fixed rays of f.
- $\Gamma = \{R_1 \cup \cdots \cup R_{d-1}\} \cup \{\text{landing points}\}$ is a graph.
- $U_1, \ldots, U_n$ connected components of $\mathbb{C} \setminus \Gamma$, are called basic regions of f.



GM Separation Theorem





Theorem (Goldberg, Milnor 1993)

Each basic region U_i contains exactly one interior fixed point or a virtual fixed point (parabolic invariant attracting petal) and at least one critical point.

L. Goldberg and J. Milnor. *Fixed points of polynomial maps. Part II. Fixed point portraits*. Ann. Sci. Ecole Norm. Sup. **26** (1993), 51-98.

Some corollaries

1. Two periodic stable components can always be separated by a pair of periodic rays landing together.
2. In particular, every parabolic periodic point is the landing point of n periodic rays, separating each one of the parabolic basins attached to it.
3. There are no Cremer periodic points on the boundary of Siegel disks (or other stable components).

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The proof

The proof of the GM separation theorem uses mainly the following:

- The behavior of P at ∞ ($z \mapsto z^d$). ☹
- Finiteness of the global degree (a final counting takes care of the parabolic basins). ☹
- Lefschetz fixed point theory.
- A weakly polynomial-like map (pol-like map with boundary contact) of degree d has $d-1$ critical points and d fixed points (counting the boundary fixed points). Observation: The boundary fixed points need to be repelling.

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Generalization to ETF

The generalization of the GM separation theorem to ETF encounters some difficulties.

- We do not have finite degree i.e., no counting.
- At infinity, the map is not $z \mapsto z^d$.
- There are infinitely many fixed rays and infinitely many fixed points (in general).

Theorem (Separation Theorem for ETF)

Suppose f is an entire transcendental map of finite order in PSB. Then, there are finitely many basic regions and every basic region contains exactly one interior fixed point or a parabolic invariant attracting petal.

▸ Sketch of the proof

A. M. Benini, N. Fagella. *A separation theorem for entire transcendental maps.* Proc. Lon. Math. Soc. **110** (2015), 291-324.

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4. Corollaries

Let $f \in PSB$, of finite order.

1. Two periodic Fatou components can always be separated by a pair of periodic rays landing together. In particular, every parabolic periodic point is that landing point of n periodic rays, separating each one of the parabolic basins attached to it.
2. There are no Cremer periodic points on the boundary of Siegel disks (or other stable components)
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Singular values and periodic cycles (Revisited)

But we can actually prove more.

Theorem (Benini, F. (2018))

Let f be a polynomial or an entire transcendental map of finite type, postsingularly bounded.

1. *Every basic region U whose interior fixed point is non-repelling (or* naked *repelling) contains at least one singular value s whose orbit $f^n(s)$ is contained in U for all $n \geq 0$.*
2. *Each non-repelling cycle (or naked repelling cycle) has at least one associated singular value whose singular orbit does not accumulate on any other non-repelling cycle.*

Note that the Fatou-Shishikura inequality for transcendental maps follows automatically.

Singular values and periodic cycles (Revisited)

1. The proof is based on the classical normal families argument, that shows that the boundary of a Siegel disk (for example) is accumulated by postsingular set.
2. The key point is to observe that the basic region has an invariant boundary.
3. It is not a perturbative argument and it does not use quasiconformal surgery.

We can also say something in the case of infinitely many singular values.

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Singular values and periodic cycles (Revisited)

Sketch of the proof - simple case of a fixed point

- Let B be a basic region for f with a Siegel interior fixed point.
- Observe that ∂B is invariant. Hence for all $U \in B$, cc of $f^{-1}(U)$ are either in B or outside B.
- Let Δ be the Siegel disk and $w \in \partial \Delta$.
- We take inverse images mapping Δ to Δ.

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1. Fix U_0 nbd of w. Then, $\exists n_0 \geq 0$ such that $f^{-n_0}(U) \subset \Delta$ contains a singular value s_0. Therefore $s_0 \in B$ and $f^k(s_0) \in B$ for all $k \leq n_0$. In particular $v_0 = f^{n_0}(s_0) \in U_0$.

2. Take $U_1 \subset U_0$, nbd of zw, such that $v_0 \notin U_1$. Then, $\exists n_1 \geq n_0$ such that $f^{-n_1}(U) \subset \Delta$ contains a singular value s_1. Therefore $s_1 \in B$ and $f^k(s_1) \in B$ for all $k \leq n_1$. In particular $v_1 = f^{n_1}(s_1) \in U_0$.

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Thank you for your attention!

Sketch of the proof

Step 1 Location of interior fixed points: realize that interior points belong to a reduced part of the plane where the "degree" is finite. This gives the finiteness of basic regions.

Step 2 Do an appropriate cutting of the plane and use the argument principle applied to $f(z) - z$ to find fixed points, in the absence of true polynomial-like maps.

Step 3 Compute the relevant indices using homotopies.

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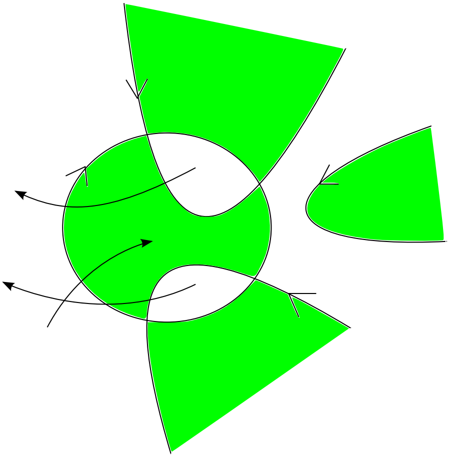
Sketch of the proof

Location of fixed points All f.p. are in $L = (D \cup \mathcal{T}) \setminus (D \cap \mathcal{T})$.

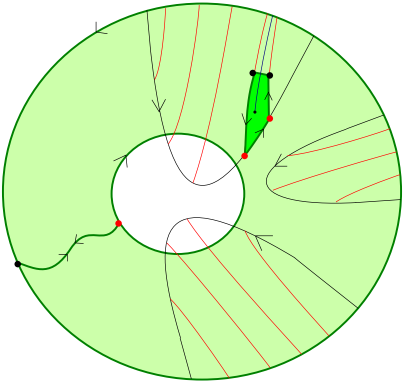


But even more.....

Lemma

If a f.d. F satisfies $F \cap D = \emptyset$, then the unique hair R_F is entirely contained in F and lands at a fixed point, which is the only fixed point in F.

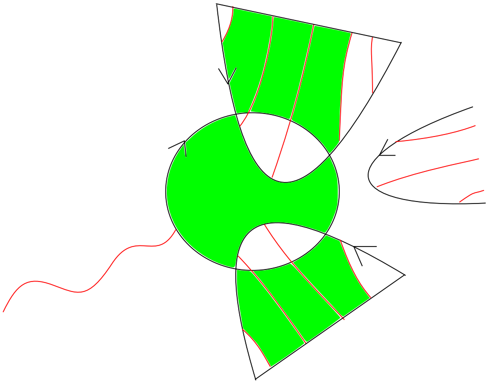
Proof: Expansion + Schwarz lemma.



Preliminaries: Reduction to finite degree.

As corollaries we obtain:

Location of interior fixed points Let $\mathcal{F}'$ be the set of fundamental domains that intersect D. All interior fixed points belong to $L' = (D \cup \mathcal{F}') \setminus (D \cap \mathcal{F}')$.



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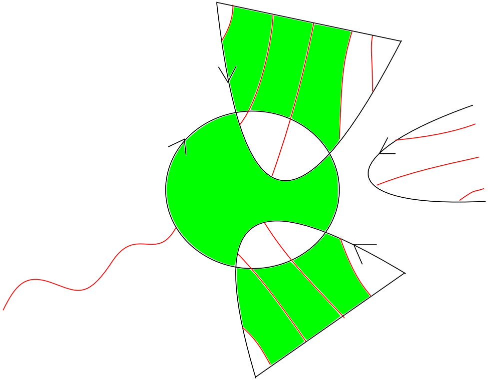

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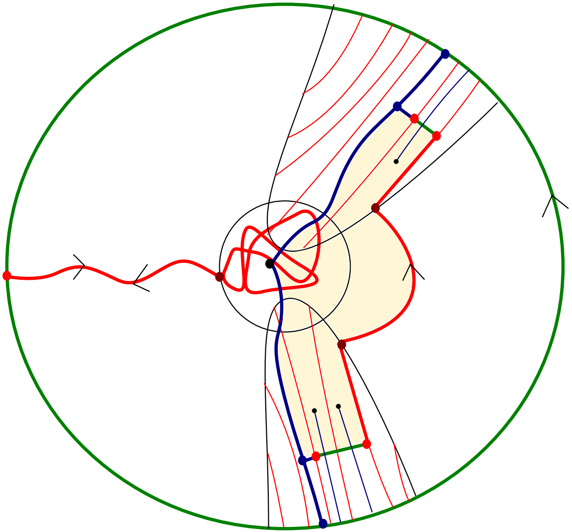
Proof in a simple case

We define a curve γ and compute $\mathrm{ind}(f(\gamma) - \gamma, 0)$.

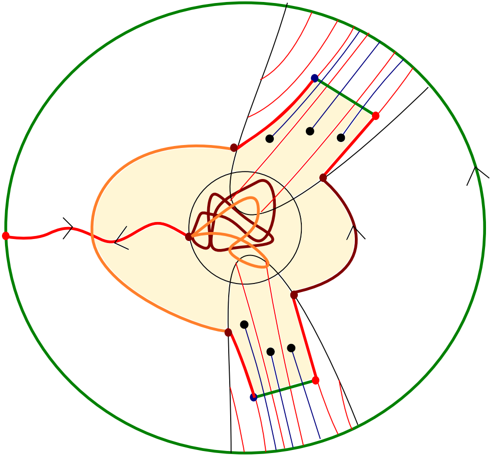


(case where the fp is repelling).

By the homotpy lemma,
index = 5 (2+1+2)
$\Rightarrow$ 1 interior f.p.

Other cases: one single basic region

The case when all hairs land alone needs some extra work (2 homotopies).



index = 7 (3+3+1)
$\Rightarrow$ 1 interior f.p.

In fact, if r fundamental domains intersect D we obtain

$$\text{index} = r + 1.$$

Go back

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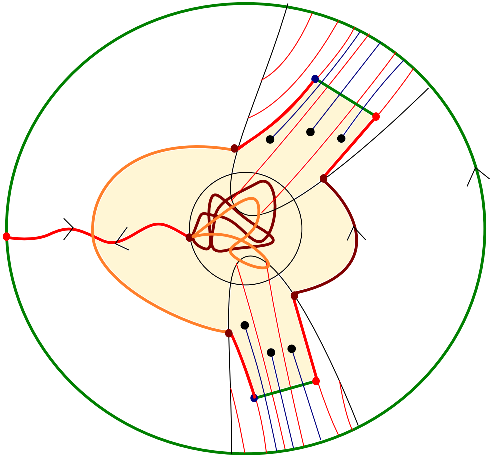

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