Bohr's inequality for harmonic mappings

Stavros Evdoridis
Dept. of Mathematics and Systems Analysis
Aalto University

New Developments in Complex Analysis and Function Theory
University of Crete - July 2, 2018

Classical Inequality

In 1914, the Danish mathematician Harald Bohr (1887-1951) proved the following theorem:

Theorem 1 (Bohr's Inequality)

Let f be an analytic function on the unit disc $\mathbb{D}$, with the Taylor expansion $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $|f(z)| < 1$. Then

$$\sum_{k=0}^{\infty} |a_k| r^k \leq 1, \quad \textit{for} \quad |z| = r \leq 1/3$$

and the constant $1/3$ is sharp.

Classical Inequality

H. Bohr proved the above theorem for $r \leq 1/6$. Right after his proof, M. Riesz, I. Schur and N. Wiener worked independently to show that it remains true for $r \leq 1/3$ and this number cannot be improved.
The best constant r for which the inequality holds, is called the *Bohr radius*.

Improvements

Recently, I. Kayumov and S. Ponnusamy showed that the classical theorem can be further improved by adding a suitable non-negative term at the left hand side of the inequality. Hence, they obtained the following results:

Theorem 2

Suppose that $f(z)=\sum_{k=0}^{\infty} a_k z^k$ is analytic in $\mathbb{D}$, $|f(z)| \le 1$ in $\mathbb{D}$ and S_r denotes the area of the image of the subdisc $|z|<r$ under the mapping f. Then

$$\sum_{k=0}^{\infty} |a_k| r^k + \frac{16}{9}\left(\frac{S_r}{\pi}\right) \le 1, \quad \text{for} \quad r \le \frac{1}{3}$$

and the constants $1/3$ and $16/9$ cannot be improved.

Improvements

Theorem 3

Suppose that $f(z) = \sum_{k=0}^{\infty} a_k z^k$ *is analytic in* $\mathbb{D}$ *and* $|f(z)| \leq 1$ *in* $\mathbb{D}$. *Then we have*

$$|a_0| + \sum_{k=1}^{\infty} \left(|a_k| + \frac{1}{2}|a_k|^2 \right) r^k \leq 1, \quad \text{for} \quad r \leq \frac{1}{3}$$

and the constants $1/3$ *and* $1/2$ *cannot be improved.*

Harmonic Mappings

Definition 1

A complex-valued function $f(z) = u(z) + iv(z)$, defined in the unit disc, is said to be a **harmonic mapping** in $\mathbb{D}$, if both u, v are real harmonic functions in $\mathbb{D}$. Then we write $\Delta f = 0$, where Δ is the complex Laplacian operator

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \overline{z}}.$$

Harmonic Mappings

If f is a harmonic mapping in $\mathbb{D}$, then it has the **canonical representation** $f = h + \overline{g}$, where h and g are analytic in $\mathbb{D}$ with $h(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=1}^{\infty} b_k z^k$.
Let us consider the affine map $f(z) = a + bz + \overline{cz}$, restricted in $\mathbb{D}$.
We can easily see that $f_{z\overline{z}} = 0$ which implies that f is a harmonic mapping with $h(z) = a + bz$ and $g(z) = cz$.

Harmonic Analogues

In 2010, Y. Abu-Muhanna proved an analogue of Bohr's inequality, for harmonic mappings.

Theorem 4

Let $f(z) = h(z) + \overline{g(z)} = \sum_{k=0}^{\infty} a_k z^k + \overline{\sum_{k=1}^{\infty} b_k z^k}$ be a harmonic mapping in $\mathbb{D}$, with $|f(z)| < 1$ for all $z \in \mathbb{D}$. Then,

$$\sum_{k=1}^{\infty} (|a_k| + |b_k|)\, r^k \leq \frac{2}{\pi}, \quad \textit{for} \quad r \leq 1/3.$$

Harmonic Analogues

Another analogue was obtained by I. Kayumov, S. Ponnusamy and N. Shakirov

Theorem 5

Suppose that $f(z) = h(z) + \overline{g(z)} = \sum_{k=0}^{\infty} a_k z^k + \overline{\sum_{k=1}^{\infty} b_k z^k}$ *is a harmonic mapping in* $\mathbb{D}$*, with* $|h(z)| \leq 1$ *and* $|g'(z)| \leq |h'(z)|$ *for all* $z \in \mathbb{D}$*. Then,*

$$|a_0| + \sum_{k=1}^{\infty} (|a_k| + |b_k|) r^k \leq 1, \quad \text{for} \quad r \leq 1/5$$

and the inequality is sharp.

Harmonic Analogues

QUESTION: Can we find $0 < r_0 < 1$ such that for any harmonic mapping f, with $|f(z)| < 1$ in $\mathbb{D}$,

$$|a_0| + \sum_{k=1}^{\infty} |a_k + b_k| r^k \leq 1,$$

for all $r < r_0$?
ANSWER: **NO**

Harmonic Analogues

Example 1

We consider a real-valued harmonic mapping f in the unit disc, such that $f(0) = 0$ and $|f(z)| < 1$ in $\mathbb{D}$. Then, $f(z) = \sum_{n=1}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{a_n z^n}$. The mapping

$$F_\beta(z) = f(z)\sin\beta + i\cos\beta, \quad \beta \in \mathbb{R}$$

is still harmonic, with $|F_\beta(z)| < 1$ in $\mathbb{D}$. If $a_n \neq 0$, then

$$\frac{|\sin\beta||a_n + a_n|}{1 - |\cos\beta|} \to \infty \quad \text{as} \quad \beta \to 0.$$

Hence, there is not any $r > 0$ such that

$$|i\cos\beta| + \sum_{n=1}^{\infty} |\sin\beta(a_n + a_n)| r^n \leq 1.$$

New Results

Our goal is to improve Theorem 5 and obtain sharp results.

Theorem 6

Suppose that $f(z) = h(z) + \overline{g(z)} = \sum_{k=0}^{\infty} a_k z^k + \overline{\sum_{k=1}^{\infty} b_k z^k}$ *is a harmonic mapping in* $\mathbb{D}$, *with* $|h(z)| \leq 1$ *and* $|g'(z)| \leq |h'(z)|$ *for* $z \in \mathbb{D}$. *Then*

$$|a_0| + \sum_{k=1}^{\infty} \left(|a_k| + |b_k|\right) r^k + \frac{3}{8} \sum_{k=1}^{\infty} \left(|a_k|^2 + |b_k|^2\right) r^k \leq 1, \quad \text{for} \quad r \leq \frac{1}{5}.$$

The constants $3/8$ *and* $1/5$ *cannot be improved.*

New Results

Theorem 7

Suppose that $f(z) = h(z) + \overline{g(z)} = \sum_{k=0}^{\infty} a_k z^k + \overline{\sum_{k=1}^{\infty} b_k z^k}$ *is a harmonic mapping in* $\mathbb{D}$*, where* $|h(z)| \leq 1$ *and* $|g'(z)| \leq |h'(z)|$ *for* $z \in \mathbb{D}$*. Then*

$$|a_0| + \sum_{k=1}^{\infty}(|a_k| + |b_k|)r^k + |h(z) - a_0|^2 \leq 1, \quad \text{for} \quad r \leq \frac{1}{5}.$$

The constant $1/5$ *is best possible.*

New Results

Theorem 8

Suppose that $f(z) = h(z) + \overline{g(z)} = \sum_{k=0}^{\infty} a_k z^k + \overline{\sum_{k=1}^{\infty} b_k z^k}$ *is a harmonic mapping of the unit disc, where* h *is a bounded function in* $\mathbb{D}$ *such that* $|h(z)| < 1$ *and* $|g'(z)| \le |h'(z)|$ *for* $z \in \mathbb{D}$. *If* S_r *denotes the area of the image of the subdisc* $|z| < r$ *under the mapping* f, *then*

$$H(r) := |a_0| + \sum_{k=1}^{\infty}(|a_k| + |b_k|)r^k + \frac{108}{25}\left(\frac{S_r}{\pi}\right) \le 1, \quad \text{for} \quad r \le \frac{1}{5}$$

and the constants $1/5$ *and* $c = 108/25$ *cannot be improved.*

Proof (Sketch)

First of all, we need sufficient upper bounds for the terms in the left hand side of the inequality.
Since h is analytic and bounded by 1 in the unit disc, it holds that

$$|a_k| \leq 1 - |a_0|^2, \quad k > 0,$$

which implies that

$$\frac{S_r}{\pi} \leq (1 - |a_0|^2)\frac{r^2}{(1-r^2)^2}, \quad r \in (0,1).$$

In addition, for the functions h and g the following inequalities are true

$$\sum_{k=1}^{\infty} |a_k| r^k \leq \begin{cases} r\dfrac{1-|a_0|^2}{1-r|a_0|} =: A(r) & \text{for } |a_0| \geq r \\ r\dfrac{\sqrt{1-|a_0|^2}}{\sqrt{1-r^2}} =: B(r) & \text{for } |a_0| < r \end{cases}$$

and

$$\sum_{k=1}^{\infty} |b_k| r^k \leq \frac{(1-|a_0|^2)r}{\sqrt{(1-|a_0|^2 r)(1-r)}} =: C(r), \quad 0 < r \leq 1/2.$$

Proof (Sketch)

- For $1/5 \leq |a_0| < 1$ and since H is an increasing function of r, we have

$$\begin{aligned} H(r) &\leq |a_0| + A(1/5) + C(1/5) + \frac{3}{16}(1-|a_0|^2)^2 \\ &= 1 - \frac{1-|a_0|}{2(5-|a_0|)\sqrt{5-|a_0|^2}}\Phi(|a_0|), \\ &\leq 1, \end{aligned}$$

  as Φ is non-negative in $[1/5, 1)$.
- For $0 \leq |a_0| < 1/5$,

$$H(r) \leq |a_0| + B(1/5) + C(1/5) + \frac{3}{16}(1-|a_0|^2)^2 < 1.$$

For the sharpness, we use the function $f_0 = h_0 + \overline{g_0}$, where $h_0(z) = \frac{a-z}{1-\bar{a}z}$ and $g_0(z) = \lambda\,(h_0(z) - a)$, $a, \lambda \in \mathbb{D}$. Then, we can see that $H(1/5) > 1$ when $a, \lambda \to 1$ and $c > 108/25$.

THANK YOU!