MAT310 Basic Project 3: eigenvalues and eigenvectors

Due December 15, 2004

This project contains the problem about Fibonacci numbers from the project on Jordan forms. You can submit these two projects together only if they do not overlap (i.e. apart from Fibonacci problem you also solved another problem from the project on Jordan forms).

Fibonacci problem: Let a_n be the sequence of Fibonacci numbers defined by the recurrent relation:

$$a_{n+2} = a_n + a_{n+1}, \quad a_0 = 1, a_1 = 1.$$

Solve exercises below to find an explicit formula for a_n.

Exercise 1. Consider an operator A on the vector space $\mathbb{R}^2$ such that A maps the vector with coordinates (x, y) to the vector $(y, x + y)$. Show that A maps (a_{n-2}, a_{n-1}) to (a_{n-1}, a_n), where a_n is the Fibonacci sequence. Write the matrix $A_{\mathcal{E}}$ of A in the standard basis $\mathcal{E}$ and prove that

$$A_{\mathcal{E}}^n \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix}.$$

Exercise 2. Diagonalize the operator A. I.e. find its eigenvalues, eigenvectors and show that eigenvectors form a basis in $\mathbb{R}^2$. Denote this basis by $\mathcal{B}$. Write a matrix $A_{\mathcal{B}}$ of A in the basis $\mathcal{B}$. Find the transition matrix P from $\mathcal{E}$ to $\mathcal{B}$.

Exercise 3. Find $A_{\mathcal{B}}^n$. Then argue that $A_{\mathcal{E}}^n = P^{-1} A_{\mathcal{B}}^n P$ using the relation $A_{\mathcal{E}} = P^{-1} A_{\mathcal{B}} P$. Find $A_{\mathcal{E}}^n$ and write an explicit formula for the n-th Fibonacci number.