Mat 310 – Linear Algebra – Fall 2004

Basic Project 2

Determinants

Due December 15

In the problems below all matrices are square and have either complex or real coefficients. In your solutions, you can use various properties of the determinant. Most of them are listed in *Summary1.pdf* on the main course page. Another useful property is as follows.

(1) If the matrix A' is obtained from a matrix A by an elementary row operation of the first type (i.e. row R_i gets replaced with the row $R_i + cR_j$), then

$$\det\,(A') = \det\,(A).$$

(2) If the matrix A' is obtained from a matrix A by an elementary row operation of the second type (i.e. row R_i gets replaced with the row cR_i for some nonzero c), then

$$\det\,(A') = c\,\det\,(A).$$

Of course, the same holds for column operations, since $\det(A) = \det(A^t)$.

1. Find the determinant of a matrix

$$(a)\begin{pmatrix} 2004 & 2005 & 2006 \\ 2004 & 2006 & 2007 \\ 2004 & 2007 & 2009 \end{pmatrix} \quad (b)\begin{pmatrix} 3 & 9 & 27 \\ 4 & 16 & 64 \\ 5 & 25 & 125 \end{pmatrix} \quad (c)\begin{pmatrix} cos\phi & sin\phi \\ -sin\phi & cos\phi \end{pmatrix}.$$

2. Define the area of a parallelogram $ABCD$ as the product of the lengths of the sides AB and AD times the sine of the angle BAD. Prove that this area equals to the absolute value of the determinant of the 2×2 matrix whose first and second rows consist of the coordinates of the vectors AB and AD relative to the standard basis. What can you say about the sign of such determinant (e.g. when is it positive)?

3. Let $v_1, \ldots, v_n$ be n vectors in the n-tuple space (so that each vector is an n-tuple). Show that $v_1, \ldots, v_n$ are linearly dependent if and only if the determinant of the matrix with the rows $v_1, \ldots, v_n$ is zero. Use this to prove that there exists a unique function D defined on collections of n vectors in the n-tuple space such that

(1) The value of D on $v_1, \ldots, v_n$ is zero whenever $v_1, \ldots, v_n$ are linearly dependent.

(2) The value of D on the standard basis vectors $e_1, \ldots, e_n$ is 1.

4. Prove that the determinant of a *skew-symmetric* 4×4 matrix

$$\begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix}$$

is a perfect square (i.e. it is a square of some polynomial of degree 2 in a, b, c, d, e, f).

5. Let A be an $n \times n$ matrix with real entries. Consider the function $f : \mathbb{R} \to \mathbb{R}$ such that $f(c) = \det(I + cA)$. Prove that the value $f'(0)$ of the derivative of f equals the trace of A. Then show that for small values of c the linear function $f_0(c) = 1 + \text{trace}(A)c$ is a good approximation for the function $f(c)$.

6. Find the determinant of the Vandermonde matrix

$$\begin{pmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{pmatrix}.$$