Basic Project 1
Vector spaces and operators
Due December 15

1. Let $W_1$ and $W_2$ be linear subspaces of a vector space $V$. Prove that the following three conditions are equivalent.
   (1) $W_1 + W_2 = V$ and $W_1 \cap W_2 = \{0\}$.
   (2) For each vector $\alpha \in V$ there are unique vectors $\alpha_1 \in W_1$ and $\alpha_2 \in W_2$ such that $\alpha = \alpha_1 + \alpha_2$.
   (3) There exists a basis in $V$ such that each vector in this basis belongs either to $W_1$ or to $W_2$.
2. Consider the vectors in $\mathbb{R}^4$ defined by
   $$\alpha_1 = (1, 0, 1, 1), \quad \alpha_2 = (1, 0, 2, 1), \quad \alpha_3 = (1, 2, 0, 1), \quad \alpha_4 = (3, 2, 3, 3)$$
   (a) What is the dimension of the subspace $W$ of $\mathbb{R}^4$ spanned by the four given vectors? Find a basis for $W$ and extend it to a basis $B$ of $\mathbb{R}^4$.
   (b) Use a basis $B$ of $\mathbb{R}^4$ as in (a) to characterize all linear transformations $T : \mathbb{R}^4 \to \mathbb{R}^4$ that have the same null space $W$. What can you say about the rank of such a $T$? What is therefore the precise condition on the values of $T$ on $B$?
   (c) Give an explicit example of an operator $T : \mathbb{R}^4 \to \mathbb{R}^4$ such that the range of $T$ is $W$.
3. Prove that vectors
   $$\alpha_1 = (1, 1, 1, 1), \quad \alpha_2 = (1, 1, 2, 1), \quad \alpha_3 = (0, 1, 0, 1), \quad \alpha_4 = (1, 1, 1, 0)$$
form a basis for $\mathbb{R}^4$. What are the coordinates of the vector $(a, b, c, d)$ in this basis?
4. Let $V$ be the vector space over $\mathbb{R}$ of all real polynomial functions $p$ of degree at most 2.
   (a) What are the coordinates of the polynomial function $a + bx + cx^2$ with respect to the ordered basis $\{1 - x^2, \; 1 + x + x^2, \; 1\}$ in $V$?
   (b) For any fixed $a \in \mathbb{R}$ consider the shift operator $T : V \to V$ with $(Tp)(x) = p(x+a)$. Consider also the differentiation operator $D : V \to V$ with $Dp = p'$. Find the range, null space, rank and nullity of the operators $TD$, $DT$, $D^2$ and $T^2$. Which of these operators are isomorphisms? Write down the matrices of the operators $TD$, $D^2$ and $T^2$ with respect to the ordered basis $B = \{1, x, x^2\}$.
5. Let $T$ be the linear operator on $\mathbb{R}^2$ defined by $T(x_1, x_2) = (-\frac{\sqrt{2}}{2}(x_1 + x_2), \frac{\sqrt{2}}{2}(x_1 - x_2))$.
   (a) What is the matrix of $T$ in the standard ordered basis for $\mathbb{R}^2$?
   (b) Interpret the operation of $T$ geometrically.
   (c) What is the matrix of $T$ in the ordered basis $B = \{\alpha_1, \alpha_2\}$, where $\alpha_1 = (1, 1)$ and $\alpha_2 = (2, 0)$?
   (d) Prove that for every real number $c$ the operator $(T - cI)$ is invertible.
   (e) Find all complex numbers $c$ such that the operator $(T - cI)$ is not invertible.
6. Let $T \in L(V, V)$ be an operator on the vector space $V$ with the null space $W_1$ and the range $W_2$. Suppose that $U \in L(V, V)$ is another linear operator on $V$ such that $TU = UT$. Prove that $U(W_1)$ is the subspace of $W_1$, and $U(W_2)$ is the subspace of $W_2$. 

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