Mat 310 – Linear Algebra – Fall 2004

Basic Project 1

Vector spaces and operators

Due December 15

1. Let W_1 and W_2 be linear subspaces of a vector space V. Prove that the following three conditions are equivalent.

(1) $W_1 + W_2 = V$ and $W_1 \cap W_2 = \{0\}$.

(2) For each vector $\alpha \in V$ there are *unique* vectors $\alpha_1 \in W_1$ and $\alpha_2 \in W_2$ such that $\alpha = \alpha_1 + \alpha_2$.

(3) There exists a basis in V such that each vector in this basis belongs either to W_1 or to W_2.

2. Consider the vectors in $\mathbb{R}^4$ defined by

$$\alpha_1 = (1,0,1,1), \quad \alpha_2 = (1,0,2,1), \quad \alpha_3 = (1,2,0,1) \quad \alpha_4 = (3,2,3,3)$$

(a) What is the dimension of the subspace W of $\mathbb{R}^4$ spanned by the four given vectors? Find a basis for W and extend it to a basis $\mathcal{B}$ of $\mathbb{R}^4$.

(b) Use a basis $\mathcal{B}$ of $\mathbb{R}^4$ as in (a) to characterize all linear transformations $T : \mathbb{R}^4 \to \mathbb{R}^4$ that have the same null space W. What can you say about the rank of such a T? What is therefore the precise condition on the values of T on $\mathcal{B}$?

(c) Give an explicit example of an operator $T : \mathbb{R}^4 \to \mathbb{R}^4$ such that the range of T is W.

3. Prove that vectors

$$\alpha_1 = (1,1,1,1), \quad \alpha_2 = (1,1,2,1), \quad \alpha_3 = (0,1,0,1), \quad \alpha_4 = (1,1,1,0)$$

form a basis for $\mathbb{R}^4$. What are the coordinates of the vector (a,b,c,d) in this basis?

4. Let V be the vector space over $\mathbb{R}$ of all real polynomial functions p of degree at most 2.

(a) What are the coordinates of the polynomial function $a+bx+cx^2$ with respect to the ordered basis $\{1-x^2,\ 1+x+x^2,\ 1\}$ in V?

(b) For any fixed $a \in \mathbb{R}$ consider the *shift operator* $T : V \to V$ with $(Tp)(x) = p(x+a)$. Consider also the differentiation operator $D : V \to V$ with $Dp = p'$. Find the range, null space, rank and nullity of the operators TD, DT, D^2 and T^2. Which of these operators are isomorphisms? Write down the matrices of the operators TD, D^2 and T^2 with respect to the ordered basis $\mathcal{B} = \{1, x, x^2\}$.

5. Let T be the linear operator on $\mathbb{R}^2$ defined by $T(x_1, x_2) = (-\frac{\sqrt{2}}{2}(x_1+x_2), \frac{\sqrt{2}}{2}(x_1 - x_2))$.

(a) What is the matrix of T in the standard ordered basis for $\mathbb{R}^2$?

(b) Interpret the operation of T geometrically.

(c) What is the matrix of T in the ordered basis $\mathcal{B} = \{\alpha_1, \alpha_2\}$, where $\alpha_1 = (1,1)$ and $\alpha_2 = (2,0)$?

(d) Prove that for every real number c the operator $(T - cI)$ is invertible.

(e) Find all *complex* numbers c such that the operator $(T - cI)$ is **not** invertible.

6. Let $T \in L(V,V)$ be an operator on the vector space V with the null space W_1 and the range W_2. Suppose that $U \in L(V,V)$ is another linear operator on V such that $TU = UT$. Prove that $U(W_1)$ is the subspace of W_1, and $U(W_2)$ is the subspace of W_2.