MAT310 Project 1: the Jordan form

Due December 10, 2004

The project consists of two parts.

First, study the Jordan form by reading Chapters 6 and 7 of the textbook or any other book on this subject. I recommend the following book (there are at least two copies of it in Math/Physics Library).

I.M. Gelfand, *Lectures on linear algebra.* Interscience Tracts in Pure and Applied Mathematics, No. 9. Interscience Publishers, New York-London 1961.

This book has a well-written and self-contained chapter on the Jordan form.

Second, to demonstrate your understanding of the Jordan form you should solve at least one problem and two exercises from the list below. Type write your solutions (handwritten texts are not accepted). This will be the text of your project. The number of pages in your text should not exceed 3.

Part 1

To check if you have studied the Jordan form thoroughly see if you can answer the following questions. You are not required to write the answers. This is just a check list for yourself.

- What is the Jordan form of an operator on a finite dimensional complex vector space?

  E.g. do you know what are eigenvalues, eigenvectors and characteristic polynomial of an operator? What is an elementary Jordan matrix?

- How the Jordan form can be used?

  E.g. how is the Jordan form used to solve linear differential equations (see p. 248 Example 8 in the textbook) and systems of linear differential equations (see Problems 1 and 3 from the list below). Other applications are discussed in Exercises 1,3 and Problem 2 from the list below

- Why does the Jordan form always exist?

  E.g. how to find all eigenvectors for a given eigenvalue? How to construct a basis such that the matrix of an operator in this basis is the Jordan matrix. You will need this to solve Exercise 2 from the list below.

- How to find the Jordan form of an operator?

  E.g. how to find the eigenvalues of an operator and their multiplicities? What is an algorithm for finding the sizes of elementary Jordan matrices for a given operator? You will need this for Problems 1,2,3 and Exercise 2

Part 2

Solve at least two exercises and at least one problem from this list. Type write your solutions.

Exercises:

1)(p.250 Exercise 9 from the textbook) Classify up to similarity all $n \times n$ complex matrices A such that $A^n = I$.

2) Find all eigenvectors of an operator A on $\mathbb{C}^2$ with the following matrix

$$\begin{pmatrix} 6 & -1 \\ 16 & -2 \end{pmatrix}.$$

What is its Jordan form? Give an example of a basis in $\mathbb{C}^2$ such that the matrix of A in this basis is the Jordan form of A.

3)(p. 250 Exercise 16 from the textbook) Prove that every invertible $n \times n$ matrix has a square root.

Hint: show that if A has square root, then any similar matrix PAP^{-1} also has square root. Then take the Jordan form of A and compute separately square roots for all of its Jordan elementary matrices. Note that a square root of a matrix $I + N$, where N is a nilpotent matrix (i.e. $N^k = 0$ for some k), is the value of the binomial series for $(1+t)^{\frac{1}{2}}$ at $t = N$ (show that this series converges for $t = N$).

Problems:

1) Solve the following system of linear differential equations in 4 functions y_1, y_2, y_3, y_4

$$\begin{pmatrix} y_1' \\ y_2' \\ y_3' \\ y_4' \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & -1 \\ 0 & 2 & 1 & -1 \\ 1 & -1 & 2 & -1 \\ 1 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}.$$

Hint: find the Jordan form A of the above 4×4 matrix together with the Jordan basis. Solve the system corresponding to A and use its solutions to find the solutions of the initial system.

2) Let a_n be the sequence of Fibonacci numbers defined by the recurrent relation:

$$a_{n+2} = a_n + a_{n+1}, \quad a_0 = 1, a_1 = 1.$$

Find an explicit formula for a_n.

Hint: consider an operator A on the 2-dimensional vector space V such that A maps the vector with coordinates (x, y) to the vector $(y, x + y)$. Then A maps (a_{n-2}, a_{n-1}) to (a_{n-1}, a_n), where a_n is the Fibonacci sequence. Note that any vector $v \in V$ defines a sequence whose nth and $(n + 1)$st terms are coordinates of the vector $A^n v$ and this correspondence is linear. Find the eigenvectors of A and consider the corresponding sequences.

3) Let A be an operator on a vector space. Define the exponent of A as the operator e^A obtained by taking the value of the exponential power series at A, i.e.

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \frac{1}{4!}A^4 + \dots . \qquad (*)$$

(a) Prove that if A and B commute then

$$e^{A+B} = e^A e^B.$$

(b) Show that the power series $(*)$ converges for all operators.

(c) Compute the exponent of the matrix from Exercise 2

$$\begin{pmatrix} 6 & -1 \\ 16 & -2 \end{pmatrix}.$$

Hint: First, prove that $e^A = P^{-1} e^{PAP^{-1}} P$, so that if the series converges for some matrix similar to A, then it converges for A as well. Then show that it converges for the Jordan form of an operator.