Project 1

(due 02/14/02)

Discuss the questions below in a concise and precise essay, at most 2 typed pages long. If you use a reference, quote it and do not copy. Use your own words. You might be interested how to typeset mathematics well (also physics, chemistry, or anything). There is a powerful computer language for that called TeX. Everyone having to write math papers needs to learn it sooner or later. Maybe you can pick it up along the way this semester. I am attaching a source file for this page, so you can take a first glimpse at TeX.

Consider a continuous curve $c : [\alpha, \beta] \longrightarrow \mathbf{R}^2$. For any (finite) subdivision $\tau = (t_0, t_1, ..., t_n)$ of the parameter interval, $\alpha = t_0 < t_1 < ... < t_n = \beta$, set

$$L_\sigma = \sum_{i=1}^{n} |c(t_i) - c(t_{i-1})| \ .$$

Interpret L_σ as the naive length of an inscribed polygon. Why should these numbers not exceed the intuitive arc length of c?

Let $S = \{L_\sigma\}$ be the set of all lengths of inscribed finite polygons. The curve c is called *rectifiable* if S is bounded. In that case, the least upper bound of S is called the *length* L of c, so

$$L = \sup S \ .$$

1. Explicitly describe an example of a continuous curve $c : [0, 1] \longrightarrow \mathbf{R^2}$ which is not rectifiable.

2. Carefully argue that if c is rectifiable and σ is any subdivision as above, then all the restrictions $c_i = c|[t_{i-1}, t_i]$ for $i = 1, ...n$ are rectifiable, and if L_i denotes the length of c_i, then $L = L_1 + ... + L_n$.

3. Prove in detail that if c is a C^1 curve, then c is rectifiable and we have the familiar formula

$$L = \int_\alpha^\beta |c'(t)| \, dt \ .$$

4. The curve c is *piecewise* C^1 if there is a subdivision σ so that the restrictions c_i as above are all C^1 curves (but c may have 'corners' at the time t_i, so the velocity field c' is possibly discontinuous at t_i).

Show that in this case c is still rectifiable and we have the same integral formula as before, if we interpret the integral in the usual way allowing finitely many 'jump discontinuities'.

5. Define what it means for c to be *piecewise linear*. Such curves we might want to call *polygons*. Notice that they can look very complicated, self-intersect etc. Usually one likes to think of *polygons* in a much more restricted way as being *convex* perhaps. Argue that if you parametrize the above inscribed polygons for a given subdivision σ in the natural way piecewise linearly on the interval $[\alpha, \beta]$ (how?), then their length is L_σ. Make sure you realize and fully appreciate why this is a non-trinial question!