Homework assignment 9
Due date: November 5

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Exercise 2. Let T be the linear operator from $\mathbb{R}^3$ to $\mathbb{R}^2$ defined by

$$T(x_1, x_2, x_3) = (x_1 + x_2, 2x_3 - x_1).$$

(a) If $\mathcal{B}$ is the standard ordered basis for $\mathbb{R}^3$ and $\mathcal{B}'$ is the standard ordered basis for $\mathbb{R}^2$, what is the matrix of T relative to the pair $\mathcal{B}, \mathcal{B}'$?

(b) If $\mathcal{B} = \{\alpha_1, \alpha_2, \alpha_3\}$ and $\mathcal{B}' = \{\beta_1, \beta_2\}$, where

$$\alpha_1 = (1, 0, -1), \quad \alpha_2 = (1, 1, 1), \quad \alpha_3 = (1, 0, 0), \quad \beta_1 = (0, 1), \quad \beta_2 = (1, 0)$$

what is the matrix of T relative to the pair $\mathcal{B}, \mathcal{B}'$?

Exercise 3. Let T be a linear operator on F^n, let A be the matrix of T in the standard ordered basis for F^n, and let W be the subspace of F^n spanned by the column vectors of A. What does W have to do with T?

Exercise 5. Let T be the linear operator on $\mathbb{R}^3$ whose matrix in the standard ordered basis is

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{pmatrix}.$$

Find a basis for the range of T and a basis for the null space of T.

Exercise 7. Let T be the linear operator on $\mathbb{R}^3$ defined by

$$T(x_1, x_2, x_3) = (3x_1 + x_3, -2x_1 + x_2, -x_1 + 2x_2 + 4x_3).$$

(a) What is the matrix of T in the standard ordered basis for $\mathbb{R}^3$?

(b) What is the matrix of T in the ordered basis $\{\alpha_1, \alpha_2, \alpha_3\}$ where $\alpha_1 = (1, 0, 1)$, $\alpha_2 = (-1, 2, -1)$ and $\alpha_3 = (2, 1, 1)$?

(c) Prove that T is invertible and give a rule for T^{-1} like the one that defines T.

Exercise 8. Let θ be a real number. Prove that the following two matrices are similar over the field of complex numbers:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}.$$

(*Hint:* Let T be the linear operator on $\mathbb{C}^2$ that is represented by the first matrix in the standard ordered basis. Find vectors α_1 and α_2 such that $T\alpha_1 = e^{i\theta}\alpha_1$, $T\alpha_2 = e^{-i\theta}\alpha_2$, and $\{\alpha_1, \alpha_2\}$ is a basis.)

Bonus exercise 12. Let V be an n-dimensional space over the field F, and $\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\}$ an ordered basis for V.

(a) There is a unique linear operator T on V such that

$$T\alpha_i = \alpha_{i+1}, \quad j = 1, \ldots, n-1, \quad T\alpha_n = 0.$$

What is the matrix A of T in the ordred basis $\mathcal{B}$?

(b) Prove that $T^n = 0$ but $T^{n-1} \neq 0$.

(c) Let S be any linear operator on V such that $S^n = 0$ but $S^{n-1} \neq 0$. Prove that there is an ordered basis $\mathcal{B}'$ for V such that the matrix of S in the ordered basis $\mathcal{B}'$ is the matrix A of part (a).

(d) Prove that if M and N are $n \times n$ matrices over F such that $M^n = N^n = 0$ but $M^{n-1} \neq 0 \neq N^{n-1}$, then M and N are similar.

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Exercise 2. Let $\mathcal{B} = \{\alpha_1, \alpha_2, \alpha_3\}$ be the basis for $\mathbb{C}^3$ defined by

$$\alpha_1 = (1, 0, -1) \quad \alpha_2 = (1, 1, 1) \quad \alpha_3 = (2, 2, 0).$$

Find he dual basis of $\mathcal{B}$.

Exercise 3. If A and B are $n \times n$ matrices over the field F, show that trace(AB) =trace(BA). Now show that similar matrices have the same trace.

Exercise 4. Let V be the vector space of all polynomial functions p from $\mathbb{R}$ to $\mathbb{R}$ that have degree 2 or less:

$$p(x) = c_0 + c_1 x + c_2 x^2.$$

Define three linear functionals on V by

$$f_1(p) = \int_0^1 p(x)dx, \quad f_2(p) = \int_0^2 p(x)dx, \quad f_3(p) = \int_0^{-1} p(x)dx.$$

Show that $\{f_1, f_2, f_3\}$ is a basis for V^* by exhibiting the basis for V of which it is dual.

Exercise 8. Let W be the subspace of $\mathbb{R}^5$ that is spanned by the vectors

$$\alpha_1 = e_1 + 2e_2 + e_3, \quad \alpha_2 = e_2 + 3e_3 + 3e_4 + e_5,$$

$$\alpha_3 = e_1 + 4e_2 + 6e_3 + 4e_4 + e_5.$$

Find a basis for W^0.

Exercise 11. Let W_1 and W_2 be the subspaces of a finite-dimensional vector space V.

(a) Prove that $(W_1 + W_2)^0 = W_1^0 \cap W_2^0$.

(b) Prove that $(W_1 \cap W_2)^0 = W_1^0 + W_2^0$.

Bonus exercise 14. Let F be a field of characteristic 0, and let V be a finite-dimensional vector space over F. If $\alpha_1, \ldots, \alpha_m$ are finitely many vectors in V, each different from the zero vector, prove that there is a linear functional f on V such that

$$f(\alpha_i) \neq 0, \quad i = 1, \ldots, m.$$