Homework assignment 8
Due date: October 29

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Exercise 1. Let T and U be the linear operators on $\mathbb{R}^2$ defined by

$$T(x_1, x_2) = (x_2, x_1) \text{ and } U(x_1, x_2) = (x_1, 0).$$

(a) How would you describe T and U geometrically?

(b) Give rules like the ones defining T and U for each of the transformations $(U + T)$, UT, TU, T^2, U^2.

Exercise 3. Let T be the linear operator on $\mathbb{R}^3$ defined by

$$T(x_1, x_2, x_3) = (3x_1, x_1 - x_2, 2x_1 + x_2 + x_3).$$

Is T invertible? If so, find a rule for T^{-1} like the one that defines T.

Exercise 4. For the linear operator of Exercise 3, prove that

$$(T^2 - I)(T - 3I) = 0.$$

Exercise 7. Find two linear operators T and U on $\mathbb{R}^2$ such that $TU = 0$ but $UT \neq 0$.

Exercise 9. Let T be a linear operator on the finite-dimensional space V. Suppose there is a linear operator U on V such that $TU = I$. Prove that T is invertible and $U = T^{-1}$. Show that this is false when V is not finite-dimensional. (*Hint:* Let $T = D$ be the differentiation operator on the space of polynomials.)

Bonus exercise 11. Let V be a finite-dimensional vector space and let T be a linear operator on V. Suppose that $\operatorname{rank}(T^2) = \operatorname{rank}(T)$. Prove that the range and null space of T are disjoint, i.e. have only the zero vector in common.

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Exercise 2. Let V be a vector space over the field of complex numbers, and suppose there is an isomorphism T of V onto $\mathbb{C}^3$. Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be vectors in V such that

$$T\alpha_1 = (1, 0, i), \quad T\alpha_2 = (-2, 1 + i, 0),$$
$$T\alpha_3 = (-1, 1, 1), \quad T\alpha_4 = (\sqrt{2}, i, 3).$$

(a) Is α_1 in the subspace spanned by α_2 and α_3?

(b) Let W_1 be the subspace spanned by α_1 and α_2, and let W_2 be the subspace spanned by α_3 and α_4. What is the intersection of W_1 and W_2?

(c) Find a basis for the subspace of V spanned by the four vectors α_j.

Exercise 4. Show that $F^{m\times n}$ (the space of $m \times n$ matrices) is isomorphic to F^{mn} (the mn-tuple space).

Bonus exercise 7. Let V and W be vector spaces over the field F and let U be an isomorphism of V onto W. Prove that $T \to UTU^{-1}$ is an isomorphism of $L(V, V)$ onto $L(W, W)$ (here $L(V, V)$ is the space of all linear operators from V to V).