Homework assignment 4
Due date: October 8

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Exercise 1. Which of the following sets of vectors $\alpha = (a_1, \ldots, a_n) \in \mathbb{R}^n$ are subspaces of $\mathbb{R}^n$ $(n \geq 3)$?

(a) all α such that $a_1 \geq 0$;
(b) all α such that $a_1 + 3a_2 = a_3$;
(c) all α such that $a_2 = a_1^2$;
(d) all α such that $a_1 a_2 = 0$;
(e) all α such that a_2 is rational.

Exercise 2. Let V be the (real) vector space of all functions f from $\mathbb{R}$ into $\mathbb{R}$. Which of the following sets of functions are subspaces of V?

(a) all f such that $f(x^2) = f(x)^2$;
(b) all f such that $f(0) = f(1)$;
(c) all f such that $f(3) = 1 + f(-5)$;
(d) all f such that $f(-1) = 0$
(e) all f that are continuous.

Exercise 3. Is a vector $(3, -1, 0, -1)$ in the subspace of $\mathbb{R}^5$ spanned by the vectors $(2, -1, 3, 2), (-1, 1, 1, -1)$ and $(1, 1, 9, -5)$?

Exercise 4. Let W be the set of all vectors $(x_1, x_2, x_3, x_4, x_5)$ in $\mathbb{R}^5$ which satisfy

$$\begin{array}{cccccc} 2x_1 & -x_2 & +\frac{4}{3}x_3 & -x_4 & = & 0 \\ x_1 & & +\frac{2}{3}x_3 & & -x_5 = & 0 \\ 9x_1 & -3x_2 & +6x_3 & -3x_4 & -3x_5 = & 0. \end{array}$$

Find a finite set of vectors that spans W.

Exercise 7. Let W_1 and W_2 be subspaces of a vector space such that the set-theoretic union of W_1 and W_2 is also a subspace. Prove that one of the spaces W_i is contained in the other.

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Exercise 1. Prove that if two vectors are linearly dependent, one of them is a scalar multiple of the other.

Exercise 3. Find a basis for the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\alpha_1 = (1, 1, 2, 4) \qquad \alpha_2 = (2, -1, -5, 2)$$
$$\alpha_3 = (1, -1, -4, 0) \qquad \alpha_4 = (2, 1, 1, 6).$$

Exercise 6. Let V be the vector space of all 2×2 matrices over the field

F. Prove that V has dimension 4 by exhibiting a basis for V that has four elements.

Exercise 7. Let V be the vector space of Exercise 6. Let W_1 be the set of matrices of the form

$$\begin{pmatrix} x & -x \\ y & z \end{pmatrix},$$

and W_2 set of matrices of the form

$$\begin{pmatrix} a & b \\ -b & c \end{pmatrix}.$$

(a) Prove that W_1 and W_2 are subspaces of V.

(b) Find the dimensions of $W_1, W_2, W_1 + W_2$, and $W_1 \cap W_2$.

Exercise 10. Let V be a vector space over the field F. Suppose there are a finite number of vectors $v_1, \dots, v_n$ in V that span V. Prove that V is finite-dimensional.

Bonus exercise 14. Let V be the set of real numbers. Regard V as a vector space over the field of *rational* numbers, with the usual operations. Prove that this vector space is *not* finite-dimensional.