Exercise 1. Which of the following sets of vectors $\alpha = (a_1, \ldots, a_n) \in \mathbb{R}^n$ are subspaces of $\mathbb{R}^n$ ($n \geq 3$)?

(a) all $\alpha$ such that $a_1 \geq 0$;
(b) all $\alpha$ such that $a_1 + 3a_2 = a_3$;
(c) all $\alpha$ such that $a_2 = a_1^2$;
(d) all $\alpha$ such that $a_1a_2 = 0$;
(e) all $\alpha$ such that $a_2$ is rational.

Exercise 2. Let $V$ be the (real) vector space of all functions $f$ from $\mathbb{R}$ into $\mathbb{R}$. Which of the following sets of functions are subspaces of $V$?

(a) all $f$ such that $f(x^2) = f(x)^2$;
(b) all $f$ such that $f(0) = f(1)$;
(c) all $f$ such that $f(3) = 1 + f(-5)$;
(d) all $f$ such that $f(-1) = 0$;
(e) all $f$ that are continuous.

Exercise 3. Is a vector $(3, -1, 0, -1)$ in the subspace of $\mathbb{R}^5$ spanned by the vectors $(2, -1, 3, 2), (-1, 1, 1, -1)$ and $(1, 1, 9, -5)$?

Exercise 4. Let $W$ be the set of all vectors $(x_1, x_2, x_3, x_4, x_5)$ in $\mathbb{R}^5$ which satisfy

$$
\begin{align*}
2x_1 & - x_2 + \frac{4}{3}x_3 - x_4 &= 0 \\
x_1 & + \frac{2}{3}x_3 - x_5 &= 0 \\
9x_1 & - 3x_2 + 6x_3 - 3x_4 - 3x_5 &= 0.
\end{align*}
$$

Find a finite set of vectors that spans $W$.

Exercise 7. Let $W_1$ and $W_2$ be subspaces of a vector space such that the set-theoretic union of $W_1$ and $W_2$ is also a subspace. Prove that one of the spaces $W_i$ is contained in the other.

Exercise 1. Prove that if two vectors are linearly dependent, one of them is a scalar multiple of the other.

Exercise 3. Find a basis for the subspace of $\mathbb{R}^4$ spanned by the vectors

$$
\alpha_1 = (1, 1, 2, 4) \quad \alpha_2 = (2, -1, -5, 2) \\
\alpha_3 = (1, -1, -4, 0) \quad \alpha_4 = (2, 1, 1, 6).
$$

Exercise 6. Let $V$ be the vector space of all $2 \times 2$ matrices over the field
Prove that $V$ has dimension 4 by exhibiting a basis for $V$ that has four elements.

**Exercise 7.** Let $V$ be the vector space of Exercise 6. Let $W_1$ be the set of matrices of the form

$$\begin{pmatrix} x & -x \\ y & z \end{pmatrix},$$

and $W_2$ set of matrices of the form

$$\begin{pmatrix} a & b \\ -b & c \end{pmatrix}.$$

(a) Prove that $W_1$ and $W_2$ are subspaces of $V$.
(b) Find the dimensions of $W_1$, $W_2$, $W_1 + W_2$, and $W_1 \cap W_2$.

**Exercise 10.** Let $V$ be a vector space over the field $F$. Suppose there are a finite number of vectors $v_1, \ldots, v_n$ in $V$ that span $V$. Prove that $V$ is finite-dimensional.

**Bonus exercise 14.** Let $V$ be the set of real numbers. Regard $V$ as a vector space over the field of *rational* numbers, with the usual operations. Prove that this vector space is *not* finite-dimensional.