Homework assignment 11
Due date: November 17

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In all exercises below, matrices have either real or complex coefficients, unless stated otherwis. By definition, the determinant of a 2×2 matrix A is the function $\det(A) = A_{11}A_{22} - A_{12}A_{21}$.

Exercise 2. Verify directly that the three functions E_1, E_2, E_3 on 3×3 matrices defined by

$$E_1(A) = A_{11}\det\begin{pmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{pmatrix} - A_{21}\det\begin{pmatrix} A_{12} & A_{13} \\ A_{32} & A_{33} \end{pmatrix} + A_{31}\det\begin{pmatrix} A_{12} & A_{13} \\ A_{22} & A_{23} \end{pmatrix},$$

$$E_2(A) = A_{12}\det\begin{pmatrix} A_{21} & A_{23} \\ A_{31} & A_{33} \end{pmatrix} - A_{22}\det\begin{pmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{pmatrix} + A_{32}\det\begin{pmatrix} A_{11} & A_{13} \\ A_{21} & A_{23} \end{pmatrix},$$

$$E_3(A) = A_{12}\det\begin{pmatrix} A_{21} & A_{22} \\ A_{31} & A_{32} \end{pmatrix} - A_{23}\det\begin{pmatrix} A_{11} & A_{12} \\ A_{31} & A_{32} \end{pmatrix} + A_{33}\det\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

are identical.

Exercise 4. Let A be a 2×2 matrix. Show that A is invertible if and only if $\det(A) \neq 0$. When A is invertible give a formula for A^{-1}.

Exercise 9. Let D be an alternating n-linear function on $n \times n$ matrices. Show that

(a) $D(A) = 0$, if one of the rows of A is 0.

(b) $D(B) = D(A)$, if B is obtained from A by adding a scalar multiple of one row of A to another.

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Exercise 1. Let A be a *skew-symmetric* 3×3 matrix, i.e.

$$A = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}.$$

Show that $\det(A) = 0$.

Exercise 2. Prove that the determinant of the Vandermonde matrix

$$A = \begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix}$$

is $(b-a)(c-a)(c-b)$.

Exercise 5. If A is an invertible $n \times n$ matrix, show that $\det(A) \neq 0$

Exercise 6. Let A be a 2×2 matrix. Prove that $\det(I + A) = 1 + \det(A)$ if and only if $\text{trace}(A) = 0$. Recall that $\text{trace}(A) = A_{11} + A_{22}$.

Exercise 7. An $n \times n$ matrix is called triangular if $A_{ij} = 0$ for $i > j$, that is, if every entry below the main diagonal is 0, or $A_{ij} = 0$ for $i < j$, that is, if every entry above the main diagonal is 0. Prove that the determinant of a triangular matrix A is the product $A_{11}A_{22} \dots A_{nn}$ of its diagonal entries.

Bonus exercise 8. Let A be a 3×3 matrix with complex entries. We form a matrix $xI - A$ with polynomial entries

$$xI - A = \begin{pmatrix} x - A_{11} & -A_{12} & -A_{13} \\ -A_{21} & x - A_{22} & -A_{23} \\ -A_{31} & -A_{32} & x - A_{33} \end{pmatrix}.$$

If $f = \det(xI - A)$, show that f is a monic polynomial of degree 3. If we write

$$f(x) = (x - c_1)(x - c_2)(x - c_3)$$

with complex numbers c_1,c_2 and c_3, prove that

$$c_1 + c_2 + c_3 = \text{trace}(A) \text{ and } c_1c_2c_3 = \det(A).$$